

Pure connection formalism for gravity: Feynman rules and the graviton-graviton scattering

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Abstract

We continue to develop the pure connection formalism for gravity. We derive the Feynman rules for computing the connection correlation functions, as well as the prescription for obtaining the Minkowski space graviton scattering amplitudes from the latter. The present formalism turns out to be significantly simpler than the one based on the metric in many aspects. The most drastic difference with the usual approach is that the conformal factor of the metric, which is a source of difficulties in the metric treatment, does not propagate in the connection formulation even off-shell. This simplifies both the linearized theory and the interactions. For comparison, in our approach the complete off-shell cubic GR interaction contains just 3 terms, with only a single term relevant at tree level. This should be compared to at least a dozen terms in the metric formalism. We put the technology developed to use and compute the simplest graviton-graviton scattering amplitudes. For GR we reproduce the well-known result. For our other, distinct from GR, interacting theories of massless spin 2 particles we obtain non-zero answers for some parity-violating amplitudes. Thus, in the convention that all particles are incoming, we find that the 4 minus, as well as the 3 minus 1 plus amplitudes are zero (as in GR), but the amplitudes with 4 gravitons of positive helicity, as well as the 3 plus 1 minus amplitudes are different from zero. This serves as a good illustration of the type of parity violation present in these theories. We find that the parity-violating amplitudes are important at high energies, and that a general parity-violating member of our class of theories "likes" one helicity (negative in our conventions) more than the other in the sense that at high energies it tends to convert all present gravitons into those of negative helicity.

1 Introduction

In paper [1] one of us showed how $\Lambda \neq 0$ General Relativity (GR) can be described in the "pure connection" formalism, in which the only field present in the Lagrangian formulation of the theory is a (complexified) $SO(3)$ connection rather than the metric. Paper [2] made the first steps towards setting up the perturbation theory in this formalism, analyzing the free theory and obtaining the propagator. In addition, a large class of interacting theories of massless spin 2 particles that are distinct from GR was exhibited, making it particularly clear that GR is just a special member in a large class of gravitational theories with similar properties. Earlier works on the pure connection formalism for gravity (in the $\Lambda = 0$ setting) include [3] and [4]. Early works on distinct from GR gravitational theories with two propagating degrees of freedom include [5] and [6]. In works of the author of [6] they are referred to as "neighbours of GR". These earlier developments used formulations involving other fields in addition to a connection; we refer the reader to [7] and references therein.

A recent work [8] by the present authors gave a more systematic treatment of the linearized theory in the pure connection setting. One outcome of [8] is a realization that the reality conditions satisfied by the connection can only be properly understood for $\Lambda \neq 0$, i.e. before the Minkowski limit is taken. This paper also derived the mode decomposition of the connection into creation/annihilation operators. The mode decomposition obtained demonstrates, in particular, that the gauge-theoretic description of gravitons is parity asymmetric, because the two helicities of the graviton are treated quite differently. One can then expect that a generic gravitational theory built using this formalism is parity-violating. This expectation will be confirmed in the present paper by a direct computation of amplitudes of some parity-violating processes.

This is the second paper in a series on the pure connection formalism for gravity. Our main objective here is to derive Feynman rules, as well as spell out the rules for extracting the graviton scattering amplitudes from the connection correlation functions. To illustrate how the formalism can be used for practical computations, we compute the simplest graviton scattering amplitudes, the two-to-two graviton ones. In the next paper from the series we apply the formalism to compute more general, in particular the so-called maximally helicity violating (MHV), amplitudes with an arbitrary number of external gravitons. The MHV amplitudes turn out to be the same for any member of the family of gravitational theories that we study. An analysis of the loop amplitudes is postponed to later papers in the series.

One of the conclusions of this paper is that the gauge-theoretic approach [2] to GR (and more general gravity theories) works, in the sense that it can be used to reproduce the known GR results. We also compute the helicity-violating scattering amplitudes that are zero in GR, but are present in a general member of our family of theories, thus confirming their parity-violating nature.

However, we hope to be able to convince the reader in a much stronger statement. Thus, not only the gauge-theoretic description of GR is equivalent (at tree level) to the standard one, but it is actually a significantly more efficient tool for practical calculations than the description based on the Einstein-Hilbert action. In particular, we shall see that the free theory looks considerably simpler in the language of connections, with the gauge-fixed Lagrangian being of the $\phi \square \phi$ type, and only a set of fields in a single irreducible component of the Lorentz group propagating. This should be contrasted with the situation in the metric-based GR, where both the tracefree and the trace part of the metric perturbation propagate, with different signs in front of the corresponding kinetic terms. Further, we shall see that in the connection formalism the interaction vertices contain much smaller number of terms as compared to the metric-based description. For instance we shall see that the GR cubic interaction vertex contains just 3 terms, of which just a single term is responsible for the graviton-graviton scattering amplitude. The simplicity of this cubic GR vertex figures even more prominently in the next paper on MHV amplitudes. One finds that computations that would be very difficult in the usual metric description are made easy by the use of the gauge theoretic framework. We will see that these significant simplifications are due to the fact that the conformal mode of the graviton, which is gauge in pure gravity, does not propagate in our formalism even off-shell.

Let us also comment on parity and unitarity in the formalism that we develop. We shall see that parity of GR, while being a symmetry of the theory, is not manifest in our description, as it treats the two graviton helicities differently. Also, as was exhibited in [8], parity in our context is related to the operation of Hermitian conjugation. It is thus directly tied with the issue of the Lagrangian being Hermitian (something that is again not manifest in our approach), and therefore unitarity. In these aspects the formalism developed here is similar to the twistor description of Yang-Mills theory proposed in [9], which also makes parity and unitarity not manifest. At the moment of writing this, we do not know whether this is just an accidental similarity or there is a more deep relation between our gauge-theoretic and the twistor approaches.

Our analysis here will be for a generic point in the space of gravity theories described in [2]. Thus, the GR results will be obtained as just a special case of more general formulas. We decided against working out just the case of GR because treating a general theory comes at almost no expense. For

readers only interested in General Relativity we will always make the versions of formulas corresponding to GR explicit. However, as we shall see, embedding GR into a larger class of theories makes some of its properties more transparent.

This paper uses spinors, and in particular the spinor helicity basis for the graviton polarizations, quite heavily (in the second half of the paper). We shall see that some aspects of our gauge-theoretic formalism can only be fully appreciated if one works in spinor terms. Relevant facts about spinors are reviewed in the text. We warn the reader that gravity community notations are adopted here, and our 2-component spinors are referred to as unprimed and primed ones (not undotted and dotted). The reason for this choice is our need, at places, to use the doubly-null tetrad representation of the metric and the self-dual 2-forms. These formulas are standard in the gravity literature using spinors, see e.g. [10], and rewriting them in terms of dotted/undotted spinors would make them look awkward (at least to us). We hope that the interested particle physics oriented reader will be able to follow in spite of these choices.

We would also like to warn the reader that at places this paper becomes quite technical, as we will have to develop from scratch many results that are simply assumed in more standard treatments. This includes a prescription for how the scattering amplitudes can be extracted from the connection correlation functions, as well as a statement of the crossing symmetry. The results we get are not surprising and could have been guessed from the start (modulo some minus signs that are characteristic of our approach treating gravity *à la* theory of fermions), but we felt we needed to do the exercise to be sure about the internal self-consistency of our approach. A particularly technical part of the paper is the one dealing with the prescription of how to take the Minkowski space limit (our theory begins as a theory of interacting gravitons in de Sitter space). Readers not interested in all these technicalities can simply take the obtained Minkowski space Feynman rules as the starting point, and then follow the part of the paper that computes the graviton scattering amplitudes. For the convenience of the reader we review the Feynman rules in the Appendix, so the paper can well be read in the opposite direction, from the end to the beginning.

The organization of this paper is as follows. We start with a brief review of relevant facts from [8]. Then, in section 3 we derive a prescription for how the scattering amplitudes can be obtained from the connection correlation functions. Here we also discuss the tricky issues of taking the Minkowski limit. The outcome of this section is a practical prescription for how to do calculations. We stress that even though the theory starts as being about gravitons in de Sitter space, the final prescription works with Minkowski space quantities, and, in particular, the usual Fourier transform is available. Then in section 4 we explain how the gauge-fixing is done, and obtain the propagator. Section 5 computes the interaction terms, up to quartic interactions. Section 6 reviews the necessary spinor technology. This technology is then immediately put to use and the graviton polarization tensors are translated into the spinors language. In section 7 we translate everything in spinor terms and state the Feynman rules in their final, most useful for computational purposes form. Section 8 then computes the graviton-graviton scattering amplitudes. We conclude with a discussion.

2 Brief summary of the previous work

In this section we write down, for the convenience of the reader, formulas from [8] that we need in the present work.

2.1 The theory

The class of gravity theories that we are studying in this work is described by the following action principle [2]

$$S[A] = i \int f(F \wedge F). \quad (1)$$

Here $i = \sqrt{-1}$, and $F^i = dA^i + (1/2)\epsilon^{ijk}A^j \wedge A^k$ is the curvature of a complexified $\text{SO}(3)$ connection A^i . The function f , referred to as the defining function of the theory, is a gauge-invariant, homogeneous function of degree one mapping symmetric 3×3 matrices to (complex) numbers. The requirement that f is of degree of homogeneity one makes the quantity $f(F \wedge F)$ in (1) a well-defined 4-form that can be integrated over spacetime. As we have already mentioned, general relativity (GR) is only a special member of the class (1), and the corresponding defining function is, see [1]

$$f_{\text{GR}}(F \wedge F) = \frac{1}{16\pi G\Lambda} \left(\text{Tr} \sqrt{F \wedge F} \right)^2, \quad (2)$$

where G is the usual Newton's constant, and Λ is the cosmological constant.

2.2 Background

As is described in details in [8], we fix a background connection and expand (1) around this background. An explicit expression for the background connection can be found in [8]. Here we will only need the fact that the curvature of this connection satisfies:

$$F_{\mu\nu}^i = -M^2 \Sigma_{\mu\nu}^i, \quad (3)$$

where M is a parameter with dimensions of mass (it gives the unit in terms of which all other dimensional quantities are later measured), and $\Sigma_{\mu\nu}^i$ are the self-dual 2-forms for the de Sitter background. These 2-forms carry information about the (de Sitter) metric, and all our computations below happen in and use this background metric structure. The parameter M is then just the (inverse of) the radius of curvature of the background de Sitter metric. We shall see that special care will need to be taken when passing to the Minkowski limit $M \rightarrow 0$, as many of the intermediate formulas will contain inverse powers of M .

2.3 Convenient way to write the action

To obtain variations of (1), it is very convenient to rewrite it in a form such that the defining function is applied to a certain 3×3 matrix rather than 4-form valued 3×3 matrix. This is most conveniently achieved by introducing

$$\hat{X}^{ij} = \frac{1}{8iM^4} \epsilon^{\mu\nu\rho\sigma} F_{\mu\nu}^i F_{\rho\sigma}^j. \quad (4)$$

Here $\epsilon^{\mu\nu\rho\sigma}$ is obtained from the volume form $\epsilon_{\mu\nu\rho\sigma}$ of the de Sitter background metric, by raising all of its indices using the metric. The convenience of this choice lies in the fact that on the background (3) we have $\hat{X}^{ij} = \delta^{ij}$. The integrand in (1) can then be written in terms of \hat{X}^{ij}

$$S[A] = -2M^4 \int d^4x \sqrt{-g} f(\hat{X}), \quad (5)$$

where $\sqrt{-g}$ is the square root of the determinant of the de Sitter background metric. Now, when determining the variations of this action, the variations should only be applied to the function f and thus the curvature F on which it non-linearly depends. Thus, the variations of (5) are very easy to compute.

2.4 Variations of the matrix \hat{X}

We start by giving the variations of \hat{X} , as a function of the connection, evaluated at the background $\hat{X}^{ij} \doteq \delta^{ij}$. We have:

$$\begin{aligned}\delta \hat{X}^{ij} &\doteq -\frac{1}{M^2} \Sigma^{(i\mu\nu} D_\mu \delta A_\nu^{j)}, \\ \delta^2 \hat{X}^{ij} &\doteq \frac{1}{iM^4} \epsilon^{\mu\nu\rho\sigma} D_\mu \delta A_\nu^i D_\rho \delta A_\sigma^j - \frac{1}{M^2} \Sigma^{(i\mu\nu} \epsilon^{j)kl} \delta A_\mu^k \delta A_\nu^l, \\ \delta^3 \hat{X}^{ij} &\doteq \frac{3}{iM^4} \epsilon^{\mu\nu\rho\sigma} D_\mu \delta A_\nu^{(i} \epsilon^{j)kl} \delta A_\rho^k \delta A_\sigma^l.\end{aligned}\tag{6}$$

Finally, the fourth variation is zero $\delta^4 \hat{X}^{ij} = 0$ even away from the background. In all expressions above D_μ is the covariant derivative with respect to the background connection. Thus, it is important to keep in mind that D 's do not commute:

$$2D_{[\mu} D_{\nu]} V^i = \epsilon^{ijk} F_{\mu\nu}^j V^k, \tag{7}$$

for an arbitrary Lie algebra valued function V^i . Here $F_{\mu\nu}^i$ is the background curvature (3). Thus, the commutator (7) is of the order M^2 . This has to be kept in mind when (in the limit $M \rightarrow 0$) replacing the covariant derivatives D with the usual partial derivatives.

2.5 Variations of the action

The variations of the action (5) are now easy to compute. We have

$$\begin{aligned}\delta S &\doteq -2M^4 \int f_{ij}^{(1)} \delta \hat{X}^{ij}, \quad \delta^2 S \doteq -2M^4 \int \left[f_{ijkl}^{(2)} \delta \hat{X}^{ij} \delta \hat{X}^{kl} + f_{ij}^{(1)} \delta^2 \hat{X}^{ij} \right], \\ \delta^3 S &\doteq -2M^4 \int \left[f_{ijklmn}^{(3)} \delta \hat{X}^{ij} \delta \hat{X}^{kl} \delta \hat{X}^{mn} + 3f_{ijkl}^{(2)} \delta^2 \hat{X}^{ij} \delta \hat{X}^{kl} + f_{ij}^{(1)} \delta^3 \hat{X}^{ij} \right], \\ \delta^4 S &\doteq -2M^4 \int \left[f_{ijklmnpq}^{(4)} \delta \hat{X}^{ij} \delta \hat{X}^{kl} \delta \hat{X}^{mn} \delta \hat{X}^{pq} + 6f_{ijklmn}^{(3)} \delta^2 \hat{X}^{ij} \delta \hat{X}^{kl} \delta \hat{X}^{mn} \right. \\ &\quad \left. + 4f_{ijkl}^{(2)} \delta^3 \hat{X}^{ij} \delta \hat{X}^{kl} + 3f_{ijkl}^{(2)} \delta^2 \hat{X}^{ij} \delta^2 \hat{X}^{kl} \right].\end{aligned}\tag{8}$$

Here we have omitted the integration measure $\sqrt{-g} d^4x$ for brevity. The quantities $f_{ij\dots}^{(n)}$ are partial derivatives of the defining function with respect to its matrix-valued argument. As is explained in [8], when evaluated at the background $\hat{X}^{ij} = \delta^{ij}$, these derivatives can be parameterized as

$$\begin{aligned}f_{ij}^{(1)} &= \frac{f(\delta)}{3} \delta_{ij}, \\ f_{ijkl}^{(2)} &= -\frac{g^{(2)}}{2} P_{ijkl},\end{aligned}\tag{9}$$

$$f_{ijklmn}^{(3)} = g^{(3)} \sum_{perm} \frac{1}{3!} P_{ijab} P_{klbc} P_{mnca} + \frac{g^{(2)}}{6} (\delta_{ij} P_{klmn} + \delta_{kl} P_{ijmn} + \delta_{mn} P_{ijkl}), \tag{10}$$

$$f_{ijklmnpq}^{(4)} = -g^{(4)} \sum_{perm} \frac{1}{4!} P_{ijab} P_{klbc} P_{mncd} P_{pqda} + \tilde{g}^{(4)} \sum_{perm} \frac{1}{3} P_{ijkl} P_{mnpq} + \dots, \tag{11}$$

where the terms denoted by dots contain δ_{ij} in at least one of the "channels" and are not going to be important for us since they give rise to a contraction that is zero in view of our gauge-fixing condition, see below. Note, however, that these terms would be of importance for vertices of valency higher than 4, but we do not consider those in this work. Here

$$P_{ijkl} := \frac{1}{2} (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}) - \frac{1}{3} \delta_{ij} \delta_{kl} \tag{12}$$

is the projector on symmetric tracefree matrices. Thus, a general theory from the class (1) is parameterized by an infinite number of (dimensionless) independent coupling constants $g^{(2)}, g^{(3)}, \dots$, as well as the value $f(\delta)$ of the defining function evaluated at the background. For GR we get

$$f_{\text{GR}}(\delta) = \frac{3M_p^2}{M^2}, \quad g_{\text{GR}}^{(2)} = \frac{M_p^2}{M^2}, \quad g_{\text{GR}}^{(3)} = \frac{3M_p^2}{4M^2}, \quad g_{\text{GR}}^{(4)} = \frac{15M_p^2}{8M^2}, \quad \tilde{g}_{\text{GR}}^{(4)} = \frac{M_p^2}{8M^2}. \quad (13)$$

Here we have defined $M_p^2 = 1/16\pi G$ and chose the scale parameter M to be related to the cosmological constant as $M^2 = \Lambda/3$.

2.6 Linearized action

The linearized action is obtained as half of the second variation of the action (5). We also define the new, canonically normalized field $a_\mu^i = (\sqrt{g^{(2)}}/i)\delta A_\mu^i$. The linearized Lagrangian then reads

$$\mathcal{L}^{(2)} = -\frac{1}{2}P_{ijkl}(\Sigma^{i\mu\nu}D_\mu a_\nu^j)(\Sigma^{k\rho\sigma}D_\rho a_\sigma^l). \quad (14)$$

This should be contrasted with a significantly more complicated linearized Lagrangian in the case of the metric-based description of GR. As is explained in [11], see also below, this Lagrangian assumes the most transparent form when written in terms of spinors. It is then quite a direct generalization of the linearized Lagrangian of Yang-Mills theory. It is worth stressing that the linearized theory is the same for any member of the class (1).

2.7 Mode expansion

Once rewritten via the space plus time decomposition, the Lagrangian (14) describes two propagating polarizations of the graviton. Only the symmetric tracefree transverse part of the spatial connection a_j^i propagates, and, taking into account the connection reality conditions, one obtains the following rather non-trivial mode decomposition for a^{ij}

$$a_{ij}(t, \vec{x}) = \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left[\varepsilon_{ij}^-(k) u_k(x) a_k^- + \varepsilon_{ij}^+(k) v_k^*(x) (a_k^-)^\dagger - \varepsilon_{ij}^+(k) v_k(x) a_k^+ - \varepsilon_{ij}^-(k) u_k^*(x) (a_k^+)^\dagger \right], \quad (15)$$

where we have introduced the following polarization tensors

$$\varepsilon_{ij}^-(k) = \frac{M}{\sqrt{2\omega_k}} m_i(k) m_j(k), \quad \varepsilon_{ij}^+(k) = \frac{\sqrt{2\omega_k}}{M} \bar{m}_i(k) \bar{m}_j(k), \quad (16)$$

and the following modes

$$u_k(x) = \frac{\mathcal{H}}{M} e^{-i\omega_k t + i\vec{k}\vec{x}}, \quad v_k(x) = \frac{M}{\mathcal{H}} e^{-i\omega_k t + i\vec{k}\vec{x}} \left(1 + \frac{i\mathcal{H}}{\omega_k} - \frac{\mathcal{H}^2}{2\omega_k^2} \right). \quad (17)$$

Note that the polarization tensors $\varepsilon_{ij}^\pm(k)$ and the modes $u_k(x), v_k(x)$ are *not* complex conjugates of each other, and so the connection is not Hermitian. Instead, it satisfies a reality condition that relates the Hermitian conjugation of the connection to its second derivative, see [8].

Here we work in the de Sitter background with $ds^2 = c^2(t) (-dt^2 + \sum_i (dx^i)^2)$, and $c(t) = \mathcal{H}/M$, where the Hubble parameter is given by

$$\mathcal{H} = \frac{M}{1 - Mt}. \quad (18)$$

The rather complicated-looking time dependent functions multiplying the usual plane waves in (17) have to do with the fact that we are in the time-dependent de Sitter background. We have normalized

the mode functions in such a way that they have a well-defined limit as $M \rightarrow 0$, and in this limit they become the usual plane waves. Later we will give a prescription for how the graviton scattering calculations can effectively be done in Minkowski space. The quantity ω_k in (15), (16) is $\omega_k = |\vec{k}|$, and the vectors m^i, \bar{m}^i in (16) are \vec{k} -dependent. Suppressing this dependence to have compact formulas, these vectors satisfy

$$i\epsilon^{ijk}z_j m_k = m_i, \quad i\epsilon^{ijk}z_j \bar{m}_k = -\bar{m}_i, \quad i\epsilon^{ijk}m_j \bar{m}_k = z_i, \quad (19)$$

and $\vec{z} = \vec{k}/|\vec{k}|$ is the unit vector in the direction of the spatial momentum \vec{k} . Finally $(a_k^\pm)^\dagger, a_k^\pm$ are the creation-annihilation operators for the two helicities, which satisfy the canonical commutational relations

$$[a_k^\pm, (a_k^\pm)^\dagger] = (2\pi)^3 2\omega_k \delta^3(k - p). \quad (20)$$

3 LSZ reduction and the Minkowski limit

In this section we describe how graviton scattering amplitudes can be derived from the connection correlation functions. We will also give a detailed prescription of how the Minkowski spacetime amplitudes are extracted. We shall see that, if one is only interested in the Minkowski space graviton scattering amplitudes, then all calculations can effectively be done in Minkowski space, where the usual Fourier transform (including in the time direction) is available. However, setting up the corresponding formalism requires some care, because of the blowing up factors of $1/M$ in the interaction vertices, see below. The content of this section is new.

3.1 Creation-annihilation operators

To obtain a version of the LSZ reduction for our theory, we need expressions for the graviton creation-annihilation operators in terms of the field operator. To obtain these, let us first give expressions for the Fourier transform of the connection field operator. From (15) we get:

$$\begin{aligned} \int d^3x e^{-i\vec{k}\vec{x}} a^{ij}(t, \vec{x}) = & \frac{1}{2\omega_k} \left[m^i m^j a_k^- \frac{\mathcal{H}}{\sqrt{2}\omega_k} e^{-i\omega_k t} + m^i m^j (a_{-k}^-)^\dagger \frac{\sqrt{2}\omega_k}{\mathcal{H}} e^{i\omega_k t} \left(1 - \frac{i\mathcal{H}}{\omega_k} - \frac{\mathcal{H}^2}{2\omega_k^2} \right) \right. \\ & \left. - \bar{m}^i \bar{m}^j a_k^+ \frac{\sqrt{2}\omega_k}{\mathcal{H}} e^{-i\omega_k t} \left(1 + \frac{i\mathcal{H}}{\omega_k} - \frac{\mathcal{H}^2}{2\omega_k^2} \right) - \bar{m}^i \bar{m}^j (a_{-k}^+)^\dagger \frac{\mathcal{H}}{\sqrt{2}\omega_k} e^{i\omega_k t} \right], \end{aligned} \quad (21)$$

We have used $m^i(-k) = \bar{m}^i(k)$ in the second and fourth terms. For compactness, the k dependence of the null vectors m^i, \bar{m}^i is suppressed in the above formula, and it is assumed that they are all evaluated at the 3-vector k . We can now take the projection of the $m^i m^j$ or $\bar{m}^i \bar{m}^j$ terms, and then device an appropriate linear combination of the connection and its first time derivative to extract the creation-annihilation operators. We get

$$\begin{aligned} a_k^- &= i\varepsilon_{ij}^+(k) \int d^3x v_k^*(x) \overleftrightarrow{\partial}_t a^{ij}, & a_k^+ &= -i\varepsilon_{ij}^-(k) \int d^3x u_k^*(x) \overleftrightarrow{\partial}_t a^{ij}, \\ (a_k^-)^\dagger &= -i\varepsilon_{ij}^-(k) \int d^3x u_k(x) \overleftrightarrow{\partial}_t a^{ij}, & (a_k^+)^\dagger &= i\varepsilon_{ij}^+(k) \int d^3x v_k(x) \overleftrightarrow{\partial}_t a^{ij}, \end{aligned} \quad (22)$$

where, as usual $f \overleftrightarrow{\partial}_t g = f \partial_t g - g \partial_t f$, and $u_k(x), v_k(x)$ are the modes given by (17). Importantly, all the creation-annihilation operators are expressed solely in terms of the field a^{ij} , and the complex conjugate field never appears. Thus, it is quite non-trivial to see that e.g. $(a_k^-)^\dagger$ is the Hermitian conjugate of a_k^- . This would involve using the reality condition for the field operator a^{ij} .

As usual in the proof of the LSZ reduction formulas, see e.g. [17] Chapter 5, we now take time integrals of the time derivative of the creation-annihilation operators. These are zero in free theory, but

the corresponding expressions are used in an interacting theory to extract the scattering amplitudes. So, we have

$$\begin{aligned}
a_k^-(\infty) - a_k^-(-\infty) &= \int dt \partial_t a_k^- = i\varepsilon_{ij}^+(k) \int d^4x v_k^*(x) \bar{D} D a^{ij}, \\
a_k^+(\infty) - a_k^+(-\infty) &= \int dt \partial_t a_k^+ = -i\varepsilon_{ij}^-(k) \int d^4x u_k^*(x) \bar{D} D a^{ij}, \\
(a_k^-)^\dagger(\infty) - (a_k^-)^\dagger(-\infty) &= \int dt \partial_t (a_k^-)^\dagger = -i\varepsilon_{ij}^-(k) \int d^4x u_k(x) \bar{D} D a^{ij}, \\
(a_k^+)^\dagger(\infty) - (a_k^+)^\dagger(-\infty) &= \int dt \partial_t (a_k^+)^\dagger = i\varepsilon_{ij}^+(k) \int d^4x v_k(x) \bar{D} D a^{ij},
\end{aligned}$$

where the readers are referred to [8] for the definitions of operators D, \bar{D} . Note that on connection satisfying its free theory field equation $\bar{D} D a = 0$ all these quantities are zero. We can now use these expressions to state the rules for extracting the graviton scattering amplitudes from the interacting theory connection correlation functions.

3.2 LSZ reduction

Quantum field theory in de Sitter space is an intricate subject with many subtleties. Because the background is time-dependent, one may argue that even the very in-out S-matrix is no longer defined, see e.g. [12] for a recent nice description of the difficulties that arise (and the possible ways to handle them). However, since in this paper we are only interested in extracting the Minkowski limit results from our formalism, we can ignore all the subtleties and proceed in an exact analogy to what one does in Minkowski space.

We thus insert a set of graviton creation operators in the far past, and then a set of annihilation operators in the far future to form a graviton scattering amplitude

$$\begin{aligned}
&\langle a_{k_-}^-(\infty) \dots a_{k_+}^+(\infty) \dots | (a_{p_-}^-)^\dagger(-\infty) \dots (a_{p_+}^+)^\dagger(-\infty) \dots \rangle \\
&\equiv \langle k_- \dots k_+ \dots | p_- \dots p_+ \dots \rangle.
\end{aligned} \tag{23}$$

Here $k_- \dots$ and $k_+ \dots$ are the set of n negative and m positive helicity outgoing graviton momenta, and $p_- \dots p_+ \dots$ are the incoming momenta of n' negative and m' positive helicity gravitons. We now add the time ordering, and then express the annihilation operators in the future in terms of those in the past, and creation operators in the past in terms of those in the future via the formulas obtained in the previous subsection. This results in the following formula for the scattering amplitude

$$\begin{aligned}
\langle k_- \dots k_+ \dots | p_- \dots p_+ \dots \rangle &= i^{n+n'-m-m'} \int d^4x_- \varepsilon_{ij}^+(k_-) v_{k_-}^*(x_-) \bar{D} D \dots \tag{24} \\
&\int d^4x_+ \varepsilon_{kl}^-(k_+) u_{k_+}^*(x_+) \bar{D} D \dots \int d^4y_- \varepsilon_{mn}^-(p_-) u_{p_-}(y_-) \bar{D} D \dots \int d^4y_+ \varepsilon_{rs}^+(p_+) v_{p_+}(y_+) \bar{D} D \dots \\
&\langle T a^{ij}(x_-) \dots a^{kl}(x_+) \dots a^{mn}(y_-) \dots a^{rs}(y_+) \rangle.
\end{aligned}$$

The time-ordered correlation functions are then obtained from the functional integral via the usual perturbative expansion.

We note some unusual features of the formula (24). A similar formula can be written for extracting the amplitudes from the metric correlators. In this case, however, because the field equation satisfied by the metric perturbation is real, there is only one solution for each sign of the frequency. In other words, only one type of mode would appear in (24), together with its complex conjugate. In the case of the connection the field equation $\bar{D} D a = 0$ is complex. This has the effect that for a given sign of the frequency, e.g. positive, there are two linearly independent solutions that we denoted by $u_k(x), v_k(x)$. This is of course just a manifestation of the parity asymmetry of our formalism, because one of the modes is used for the negative helicity and the other for the positive.

3.3 Minkowski limit

Now we would like to understand how the Minkowski spacetime graviton scattering amplitudes can be extracted from the general formula (24). The limit that we would like to take can be described as follows. Every graviton (on internal or external line) is characterized by its energy ω_k , and we would like to concentrate on gravitons for which $\omega_k \gg M$. A systematic way to do this is to have all quantities that appear in our formalism expanded in powers of M/ω_k . Thus, we will need to expand the propagators, external particle wave-functions, as well as the covariant derivatives present in the interaction vertices. For the covariant derivatives this expansion is easy to carry out, because there is just a single order M term coming from the background connection. A similar expansion of the external wavefunctions and the propagators is more difficult.

The attitude we take in this paper is as follows. We will keep only the leading order terms in the expansion of all quantities in powers of M/ω_k . We will see that these are easy (but still not completely trivial) to work out. All the subleading terms are ignored. This is guaranteed to reproduce correctly the leading part of the expansion of any quantity of interest in powers of M/ω_k . If this leading part then survives in the Minkowski limit $M \rightarrow 0$, this limit gives the desired Minkowski space answer.

Effectively this prescription means that we do not need to expand the internal propagators, as well as the covariant derivatives acting on the internal lines at all. Indeed, the leading order terms in the expansion of these objects are given by their Minkowski space limits. Thus, if we follow the above prescription all the propagators become the Minkowski space ones, and all the derivatives acting on the internal lines become the usual partial derivatives. Physically, this prescription corresponds to having all interactions between the gravitons happening in a region of de Sitter space that is small enough to neglect the effects of the curved background.

Things are not so simple for the external lines. Indeed, according to (24), the propagators $(\bar{D}D)^{-1}$ on all the external lines should be amputated, the gravitons should be put on-shell and the graviton scattering amplitudes should be extracted by integrating the correlation function (with amputated propagators) against the appropriate mode functions $u_k(x), v_k(x)$. One should also take the projection on appropriate polarizations using the ε^\pm tensors. However, our external gravitons are necessarily objects in de Sitter space, which is clear e.g. from the fact that there is no limit of the graviton polarization tensors ε^\pm as $M \rightarrow 0$. This has to do with the fact that the reality condition that was imposed to obtain the creation-annihilation operator decomposition of the connection operator is becoming singular in the $M \rightarrow 0$ limit. All in all, one cannot simply take the limit $M \rightarrow 0$ for the external graviton asymptotic states as these are intrinsically de Sitter in nature.

As we said, our prescription is to keep the leading order in powers of M terms. It is thus clear that we should keep the overall powers of M in the polarization tensors ε^\pm , as well as the powers of M in the interaction vertices, see below. However, there is also the issue of the covariant derivatives acting on the external gravitons, as well as the issue of the external graviton wave-functions. The reason why we cannot simply replace the latter by the plane waves, and the covariant derivatives by the partial derivatives is that, as we shall see below, the partial derivative acting on the external graviton plane wave can produce a zero answer for a quantity that is non-zero (but e.g. of the order M) when computed with the full covariant derivative. Further, there are terms in the Lagrangian where such a covariant derivative acting on the external graviton wave-function gets multiplied with $1/M$. By neglecting the effects of the curvature we are neglecting the order M terms in the covariant derivative, and when multiplied by $1/M$ this produces a finite in the $M \rightarrow 0$ limit quantity which we would be neglecting.

We can then expand both the external wave-functions, as well as the covariant derivatives into powers of M/ω_k . This expansion will start with the Minkowski limit objects, with the first correction being of the order M/ω_k . Importantly, as it will become clear below from our description of the interaction vertices, the vertices contain at most a single derivative acting in each line, internal or external. This implies that in the expansion of the external wave functions it is sufficient to keep just

the first order terms in M/ω_k .

Let us analyze this in more details. The covariant derivative applied to some external graviton wave-function has a part that contains the time derivative ∂_t , and another part that contains the spatial covariant derivative D_i . Note that the background connection only has the spatial component, and this is why the time derivative is the usual one ∂_t . Further, in the spatial covariant derivative there is a term of the order ω_k coming from applying the partial derivative ∂_i to the plane wave $e^{i\vec{k}\vec{x}}$, as well as a term of the order M coming from the background connection. As for the external wave-functions $u_k(x), v_k(x)$, the spatial dependence of these is the usual plane-wave $e^{i\vec{k}\vec{x}}$, while the time dependence is complicated.

As we have already mentioned, in the scattering amplitude (24) we have at most one covariant derivative hitting the external graviton wave-function. Then, according to our prescription, whenever the covariant derivative of the external wave-function admits a finite non-zero limit as $M \rightarrow 0$, we can simply replace the wave function by its Minkowski limit, and the covariant derivative by the partial derivative. However, when the wave-function is such that the covariant derivative applied to it gives zero in Minkowski limit, we will modify the Minkowski space graviton plane-wave in such a way that, when acted upon by a partial derivative, this reproduces precisely the leading order M/ω_k term of the expansion of the full result.

Let us see how this works in practice. The covariant derivative that is applied to an external graviton wave-function appears in our interaction vertices only in the combination $D_{[\mu}a_{\nu]}^i$. This Lie algebra-valued two-form can be further decomposed into its self-dual and anti-self-dual parts as in (52). From the Hamiltonian analysis in [8], see formula (87) of this reference, we know that the self-dual part of $D_{[\mu}a_{\nu]}^i$ is essentially given by the action of the operator D on the spin 2 component a^{ij} of the spatial connection. This is modulo the term involving the derivative of the temporal component of the connection a_0^i (shifted by c^i , see (87) of [8]), which is set to zero in the Hamiltonian treatment. Similarly, it is clear that the anti-self-dual part of $D_{[\mu}a_{\nu]}^i$ is obtained by applying the derivative operator \bar{D} to a^{ij} , see formula (98) of [8] for the definition of both operators. Thus, we have to consider the action of D, \bar{D} on all the wave-functions that appear in (24), i.e. on $\epsilon^-(k)u_k(x)$ and $\epsilon^-(k)u_k^*(x)$, as well as on $\epsilon^+(k)v_k(x)$ and $\epsilon^+(k)v_k^*(x)$. Moreover, as we discussed above, we are only interested in the leading-order behavior of these derivatives in the limit $M \rightarrow 0$. We get

$$\begin{aligned} D\epsilon^-(k)u_k &= D\epsilon^-(k)u_k^* = 0, \\ D\epsilon^+(k)v_k &\rightarrow -2\omega_k\epsilon^+(k)e^{-i\omega_k t + i\vec{k}\vec{x}}, \quad D\epsilon^+(k)v_k^* \rightarrow 2\omega_k\epsilon^+(k)e^{i\omega_k t - i\vec{k}\vec{x}}. \end{aligned}$$

The action of the \bar{D} operator has been worked out in (135) of [8]. In the limit $M \rightarrow 0$ we can write

$$\begin{aligned} \bar{D}\epsilon^-(k)u_k &\rightarrow 2\omega_k e^{-i\omega_k t + i\vec{k}\vec{x}}\epsilon^-(k), \quad \bar{D}\epsilon^-(k)u_k^* \rightarrow -2\omega_k e^{i\omega_k t - i\vec{k}\vec{x}}\epsilon^-(k), \\ \bar{D}\epsilon^+(k)v_k &\rightarrow -\frac{M^2}{\omega_k} e^{-i\omega_k t + i\vec{k}\vec{x}}\epsilon^+(k), \quad \bar{D}\epsilon^+(k)v_k^* \rightarrow \frac{M^2}{\omega_k} e^{i\omega_k t - i\vec{k}\vec{x}}\epsilon^+(k). \end{aligned}$$

The idea now is to devise some Minkowski spacetime wave-functions that give exactly the same leading order results when the operators $\lim_{M \rightarrow 0} D = -i\partial_t + \epsilon\partial$, $\lim_{M \rightarrow 0} \bar{D} = i\partial_t + \epsilon\partial$ are applied. We also note that the limiting case operators are just the complex conjugates of each other.

For the modes involving u_k and its complex conjugate the answer is obvious — one should just take $\lim_{M \rightarrow 0} u_k$ as the corresponding wave-function. Thus, we set

$$u_k^M(x) := e^{-i\omega_k t + i\vec{k}\vec{x}}, \tag{25}$$

and use this wave-function (and its complex conjugate) instead of $u_k(x)$ and $u_k^*(x)$ every time it appears in the LSZ formula (24). The operators that act on $u_k^M(x)$ (and its complex conjugate) are the Minkowski limit ones $-i\partial_t + \epsilon\partial$ and $i\partial_t + \epsilon\partial$.

For the v_k mode the situation is more non-trivial. If we choose it to be just the ordinary plane wave we will correctly reproduce the leading order of the action of the D operator. However, the $\lim_{M \rightarrow 0} \bar{D} = i\partial_t + \epsilon\partial$ operator will give zero. The reason why one gets a non-zero answer when acting on the full wave-function $v_k(x)$ is that this has a non-trivial time-dependent factor multiplying $e^{-i\omega_k t + i\vec{k}\vec{x}}$. The idea is then to change the frequency ω_k in the plane wave to model this non-trivial time-dependence factor. This is achieved by the following plane wave

$$v_k^M(x) := e^{-i\tilde{\omega}_k t + i\vec{k}\vec{x}}, \quad (26)$$

where

$$\tilde{\omega}_k := \sqrt{\omega_k^2 - 2M^2} = \omega_k \left(1 - \frac{M^2}{\omega_k^2} + O\left(\frac{M^4}{\omega_k^4}\right) \right). \quad (27)$$

Indeed, modulo the higher order corrections this choice of the wave-function gives precisely the required

$$(i\partial_t + \epsilon\partial)\varepsilon^+(k)v_k^M(x) = -\frac{M^2}{\omega_k}\varepsilon^+(k)v_k^M(x), \quad (i\partial_t + \epsilon\partial)\varepsilon^+(k)v_k^{*M}(x) = \frac{M^2}{\omega_k}\varepsilon^+(k)v_k^{*M}(x). \quad (28)$$

At the same time, the leading order term in the result of the action of $-i\partial_t + \epsilon\partial$ on this mode is unchanged by this shift of the frequency.

Thus, the choice (25), (26) satisfies the requirement that when acted upon by the operators $-i\partial_t + \epsilon\partial$ and $i\partial_t + \epsilon\partial$ one reproduces the leading behavior of the derivatives of the wave-functions with the full time-dependence. Therefore, if we are interested in the Minkowski limit of up to the first derivatives of the wave-functions, it is sufficient to replace the full wave-functions by (25), (26), and the operators D, \bar{D} by the corresponding $M \rightarrow 0$ limit operators.

3.4 The prescription

All in all, we see that the computation of the Minkowski graviton amplitudes can be reduced to computations with the Minkowski $1/k^2$ propagator (49), the vertices obtained by replacing the co-variant derivatives with partial ones everywhere, with the polarization tensors (16), and with the wave-functions (25), (26). The rule for the wave-functions is that the on-shell condition for the negative helicity graviton is the usual one $\omega = \pm\omega_k$, while the positive helicity graviton should be taken to have a small mass $\omega = \pm\tilde{\omega}_k$, with $\tilde{\omega}_k$ given by (27). In doing the computation one keeps all the M -dependent prefactors, and at the end takes the limit $M \rightarrow 0$ (if this exists).

Let us write the above prescription in terms of a formula. Given that our Minkowski space wave-functions (25) and (26) are just the standard plane waves, we can immediately write down the momentum space LSZ prescription. Indeed, we note that the position space integrals give us the Fourier transforms of the time-ordered correlation function. As usual we will be assigning an arrow to each line in the Feynman diagram, with the arrows on incoming lines pointing towards the diagram, and the arrows on outgoing lines away from it. Then as usual the 4-momentum of each particle is to be understood as in the direction of the arrows. This prescription takes care of the factors of u_k^M, v_k^M and their complex conjugates. The factors of $\bar{D}D$ will then amputate the propagators on the external lines. This will also absorb the overall factors of i in the formula (24). However, with our conventions for the mode decomposition there are some signs that are left over. Indeed, we have $\bar{D}D = \partial_t^2 - \Delta = -\partial^\mu\partial_\mu = -\square$. At the same time we have $-\square e^{ikx} = k^2 e^{ikx}$, and so the operator $\bar{D}D$ will cancel k^2 from each external line. However, in our conventions the propagator is $1/ik^2$, and so we will have a prefactor of $(-1)^{m+m'}$ left, where m, m' are the numbers of incoming and outgoing *positive* helicity gravitons. This nicely follows the pattern that positive helicity gravitons are a source of headache in the formalism. Of course these minus signs in the amplitudes are convention dependent

and are not of any physical significance. We finally have:

$$\langle k_- \dots k_+ \dots | p_- \dots p_+ \dots \rangle = (-1)^{m+m'} (2\pi)^4 \delta^4 \left(\sum k - \sum p \right) \quad (29)$$

$$\varepsilon_{ij}^+(k_-) \varepsilon_{kl}^-(k_+) \varepsilon_{mn}^-(p_-) \varepsilon_{rs}^+(p_+) \langle T a^{ij}(k_-) \dots a^{kl}(k_+) \dots a^{mn}(p_-) \dots a^{rs}(p_+) \rangle_{\text{amp}},$$

where the momentum space amplitude is amputated from its external line propagators. We should add to this formula the prescription that the positive helicity incoming particle 4-momentum, as well as the 4-momentum of the negative helicity outgoing particle is slightly massive, see (27), while the 4-momenta of the incoming negative and outgoing positive helicity are massless $\omega = \omega_k = |k|$.

The above prescription is guaranteed to reproduce the correct leading order M -dependence. Thus, it is only consistent if the answers one gets turn out to have non-zero limit as $M \rightarrow 0$, i.e. if all the factors of M cancel out from the end result. If this does not happen this means that the quantity computed does not have a well-defined Minkowski limit, and only makes sense in the full theory in de Sitter space.

3.5 Crossing symmetry

We can now ask about an analog of the field theory crossing symmetry relation for our scattering amplitudes. We shall discuss the crossing symmetry in the Minkowski space limit only. We recall that the usual QFT crossing symmetry arises if one takes an incoming particle of momentum \vec{k} and energy ω_k , and analytically continues the amplitude to energy $-\omega_k$. The amplitude can then be interpreted as that of an outgoing anti-particle of momentum $-\vec{k}$.

Let us see what happens in our case. Since the field is electrically neutral, our particles are their own anti-particles, so we should only expect the helicity to change if we flip one graviton from the initial to the final state. That this is indeed the case is seen from our formula (29). Indeed, to make an outgoing particle incoming one should just flip the direction of the arrow on Feynman diagram external line corresponding to that particle. This will correctly continue $\omega_k \rightarrow -\omega_k$ and $\tilde{\omega}_k \rightarrow -\tilde{\omega}_k$, as well as change the sign of the corresponding 3-momentum. Note that we do not touch the helicity states. It is then clear that if we apply the crossing symmetry to a negative helicity outgoing graviton, we produce a positive helicity incoming one, and similarly if we make a positive helicity outgoing graviton to be incoming, we will get a negative helicity one. This is clear from the fact that we are projecting on the negative/positive polarization tensors for the incoming negative/positive helicity gravitons, but do the reverse projection for the outgoing ones. Since by flipping one graviton from outgoing to incoming state we always change the total number of positive helicity gravitons in the amplitude (negative becomes positive, positive becomes negative, so there is always a change in the number of total positive helicity particles), then any crossing symmetry flip always introduces a minus sign coming from $(-1)^{m+m'}$ prefactor. Such a minus sign is appropriate for fermions, and here we see it occurring for purely bosonic particles, which again signifies the analogy between our treatment and that of fermions.

Thus, we see that the crossing symmetry operates within our formalism, and we can from now on restrict our attention to all gravitons being e.g. incoming. Realistic scattering amplitudes can then be obtained from these by applying the crossing symmetry relations.

4 Gauge-fixing and the propagator

We will now derive the Feynman rules, starting with the propagator (that was already referred to in the previous section), and then finishing with the interaction vertices. The propagator in the pure connection formalism was obtained in [2]. Here we repeat the analysis, by a slightly simpler method, and add some details such as the gauge-fixed linearized action before the Minkowski limit is taken. We also derive the ghost sector Feynman rules. We first work in de Sitter space and then take the Minkowski limit, as is explained in the previous section.

4.1 Diffeomorphisms

As is discussed in more details in [2], [8], at the linearized level diffeomorphisms act as $\delta_\xi a_\mu^i = \xi^\alpha \Sigma_{\alpha\mu}^i$. It is not hard to see what this action is by decomposing the field a_μ^i into its irreducible components with respect to the action of the Lorentz group. Thus, let us introduce the following projectors

$$P_{\mu i|\nu j}^{(3,1)} := \frac{2}{3} \left(\delta_{ij} g_{\mu\nu} + \frac{1}{2} \epsilon_{ijk} \Sigma_{\mu\nu}^k \right), \quad P_{\mu i|\nu j}^{(1,1)} := \frac{1}{3} \left(\delta_{ij} g_{\mu\nu} - \epsilon_{ijk} \Sigma_{\mu\nu}^k \right). \quad (30)$$

Both act on pairs μi of a spacetime index and an internal one. The projector $P^{(3,1)}$ is on the irreducible component $S_+^3 \otimes S_-$, and $P^{(1,1)}$ is on $S_+ \otimes S_-$ in the spinor representation $S_+^2 \otimes S_+ \otimes S_-$ that the pair μi lives in. Here S_+, S_- are two 2-dimensional fundamental representations of the Lorentz group (i.e. the representation realized by unprimed and by primed 2-component spinors). The diffeomorphisms are then simply shifts of the $(1, 1)$ component. They can be completely gauge-fixed by requiring

$$a_\mu^i = \epsilon^{ijk} \Sigma_\mu^{k\nu} a_\nu^j, \quad (31)$$

or

$$P_{\mu i|\nu j}^{(1,1)} a^{\nu j} = 0. \quad (32)$$

It is important to stress that the gauge-fixing condition for the diffeomorphisms does not contain derivatives (as is appropriate for the transformation that is merely a shift of the field in some direction in the field space).

We could now add the square of this gauge-fixing condition to the action (with some gauge-fixing parameter) and make the corresponding components of the connection propagating. However, this would have the effect that some components of the connection have the $1/k^2$ propagator, while the pure diffeomorphism gauge modes have a mode-independent, algebraic propagator. This would require dealing with the two components separately, which would make the formalism very cumbersome. To avoid this, we fix the gauge (32) sharply, i.e. work in the corresponding Landau gauge. We shall later see that the gauge-fixing condition (32) is particularly transparent when expressed in spinor terms. It will simply state that everything but the completely symmetric (in spinor indices) component of the connection is zero. This condition will be then easy to impose and it will simplify the computations significantly.

On the other hand, the remaining gauge freedom, namely the usual $SO(3)$ transformations, will be gauge-fixed (in the next subsection) as in Yang-Mills theory, i.e. by adding a $D^\mu a_\mu^i$ term squared to the Lagrangian. It is interesting to note that the way the gauge is fixed in our pure connection approach is opposite to that used in e.g. the first order formulation of general relativity that possesses both the internal as well as diffeomorphism gauge symmetry, as in our case. In the first-order formulation the gauge symmetry corresponding to $SO(1, 3)$ local gauge transformations is fixed by requiring the tetrad perturbation h^{IJ} , $I, J = 0, 1, 2, 3$ to be symmetric, i.e. by projecting away some irreducible component of the field with respect to the action of the Lorentz group. The diffeomorphisms are then fixed in the usual derivative way, using the de Donder gauge. What happens in our formalism is precisely the opposite. The diffeomorphisms are fixed in a non-derivative way by projecting away some irreducible component of the field. The gauge rotations are then fixed by adding to the action the square of a term containing the derivative of the field.

4.2 Gauge-fixing the gauge rotations

We now add to the Lagrangian the gauge-fixing term

$$\mathcal{L}_{\text{gf}} = -\frac{\alpha}{2} \left(D^\mu P_{\mu i|\nu j}^{(3,1)} a^{\nu j} \right)^2, \quad (33)$$

where the projector is inserted to make the gauge-fixing condition diffeomorphism-invariant. As in [2], we could have then done the transformations in full generality, without imposing the gauge-fixing (32). We would find that (for a choice of α to be given below) the gauge-fixed linearized Lagrangian (modulo the background curvature term) is just a multiple of $a D^\mu D_\mu P^{(3,1)} a$. However, we can simplify the computation significantly by imposing the gauge condition (32) from the very beginning.

The simpler computation is as follows. First, we note that the gauge-fixed connections satisfy $\Sigma^{\mu\nu i} a_\nu^i = 0$. We can then ignore the $\delta_{ij} \delta_{kl}$ term in the projector P_{ijkl} , and write the Lagrangian (14) as

$$\mathcal{L}^{(2)} = -\frac{1}{2} \delta_{ik} \delta_{jl} (\Sigma^{(i\mu\nu} D_\mu a_\nu^j) (\Sigma^{k\rho\sigma} D_\rho a_\sigma^l). \quad (34)$$

We now use the gauge-fixing condition (32) to obtain the following identity

$$\Sigma^{i\mu\nu} D_\mu a_\nu^j - \Sigma^{j\mu\nu} D_\mu a_\nu^i = -\epsilon^{ijk} D^\mu a_\mu^k. \quad (35)$$

In other words,

$$\Sigma^{(i\mu\nu} D_\mu a_\nu^j) = \Sigma^{i\mu\nu} D_\mu a_\nu^j + \frac{1}{2} \epsilon^{ijk} D^\mu a_\mu^k. \quad (36)$$

In the derivation of (35) we have secretly extended the covariant derivative with respect to the background connection to a derivative that also acts on the spacetime indices, so that $\Sigma^{\mu\nu i}$ can be taken through the covariant derivative. Thus, from now on there is also the usual Christoffel symbol inside D_μ .

We now substitute (36) into the Lagrangian (34), and use the gauge-fixing condition once more to convert the result into a sum of just two terms:

$$\mathcal{L}^{(2)} = -2P^{+\mu\nu\rho\sigma} \delta_{ij} D_\mu a_\nu^i D_\rho a_\sigma^j + \frac{1}{4} \delta_{ab} (D^\mu a_\mu^i) (D^\nu a_\nu^j). \quad (37)$$

Here we have used the fact that

$$\Sigma^{\mu\nu i} \Sigma^{\rho\sigma i} = 4P^{+\mu\nu\rho\sigma} = g^{\mu\rho} g^{\nu\sigma} - g^{\mu\sigma} g^{\nu\rho} - i\epsilon^{\mu\nu\rho\sigma}, \quad (38)$$

which is just a multiple of the self-dual projector. We now integrate by parts in the first term, and then represent the self-dual projector as

$$4P^{+\mu\nu\rho\sigma} = g^{\mu\rho} g^{\nu\sigma} - g^{\mu\nu} g^{\rho\sigma} + 4P^{-\mu\rho\nu\sigma}, \quad (39)$$

where P^- is the anti-self-dual projector. Then in terms that are anti-symmetric in the covariant derivatives, we express the commutator of two covariant derivatives via the curvature tensors. We have

$$D_{[\mu} D_{\rho]} a_\sigma^i = -\frac{1}{2} R_{\mu\rho\sigma}{}^\alpha a_\alpha^i + \frac{1}{2} \epsilon^{ijk} F_{\mu\rho}^j a_\sigma^k, \quad (40)$$

where $R_{\mu\rho\sigma}{}^\alpha$ is the Riemann curvature. We can now use the fact that the last term here is a purely self-dual quantity, and thus drop this part as it will be multiplied by $P^{-\mu\rho\nu\sigma}$. We get:

$$4P^{-\mu\rho\nu\sigma} D_{[\mu} D_{\rho]} a_\sigma^i = -\frac{1}{2} (g^{\mu\nu} g^{\rho\sigma} - g^{\mu\sigma} g^{\rho\nu} + i\epsilon^{\mu\rho\nu\sigma}) R_{\mu\rho\sigma}{}^\alpha a_\alpha^i = R_\nu{}^\alpha a_\alpha^i, \quad (41)$$

where $R_\nu{}^\alpha := g^{\sigma\rho} R_{\sigma\nu\rho}{}^\alpha$ is the Ricci tensor. We finally get

$$\mathcal{L}^{(2)} = -\frac{1}{2} \delta_{ij} g^{\rho\sigma} D^\mu a_\rho^i D_\mu a_\sigma^j + \frac{3}{4} \delta_{ij} (D^\mu a_\mu^i) (D^\nu a_\nu^j) + \frac{1}{2} \delta_{ij} R^{\mu\nu} a_\mu^i a_\nu^j. \quad (42)$$

It is now clear that the choice $\alpha = 3/2$ gives

$$\mathcal{L}^{(2)} + \mathcal{L}_{\text{gf}} = -\frac{1}{2}\delta_{ij}g^{\rho\sigma}D^\mu a_\rho^i D_\mu a_\sigma^j + \frac{1}{2}\delta_{ij}R^{\mu\nu}a_\mu^i a_\nu^j, \quad (43)$$

which is just the scalar field Lagrangian for every component of a_μ^i (projected onto the $(3,1)$ representation by the gauge-fixing condition (32)), plus a curvature term.

We can further rewrite (43) by using the fact that the background metric is Einstein:

$$R_{\mu\nu} = 3M^2 g_{\mu\nu}. \quad (44)$$

This gives for the gauge-fixed Lagrangian (after integrating by parts)

$$\mathcal{L}^{(2)} + \mathcal{L}_{\text{gf}} = \frac{1}{2}a^{\mu i} (D^2 + 3M^2) a_\mu^i, \quad (45)$$

where $D^2 = D^\mu D_\mu$. Note the "wrong" sign in front of the "mass" term here. It is not a source of any inconsistencies, as the squared covariant derivative contains additional terms of the order M^2 , and it overall gives rise to the mode behavior described in [8]. This is of course also just the appropriate mass term for a massless spin two field in de Sitter space.

What is significant about the gauge-fixed Lagrangian (45) is that all modes appear in it with the same sign in front of their kinetic term, unlike in the metric-based description that exhibits the conformal factor problem. This is also the reason why the Lagrangian (45) gives rise to simpler Feynman rules than in the metric case. Indeed, in our case all the fields are treated uniformly, while in the metric case one often meets (e.g. in the vertices) the tracefree part of $h_{\mu\nu}$ and its trace separately, which makes computations more involved.

4.3 Minkowski space propagator

The quadratic form in the gauge-fixed Lagrangian (43) can be inverted in full generality, using the relevant de Sitter space modes to construct the associated Green's function. However, as we have already stated above, effectively we are doing all our graviton scattering calculations in Minkowski space. Then the Lagrangian (43) admits an obvious Minkowski spacetime limit

$$\mathcal{L}^{(2)} + \mathcal{L}_{\text{gf}} = -\frac{1}{2}(\partial_\mu a_\nu^i)^2. \quad (46)$$

The propagator is now easily obtained by going to the momentum space, and is obviously a multiple of the projector $P^{(3,1)}$ times $1/k^2$. To get all the factors right we introduce into the action a source term, and integrate out the connection

$$S^{(2)}[a, J] = \int \frac{d^4 k}{(2\pi)^4} \left[-\frac{1}{2}a^{\mu i}(-k)k^2 a_\mu^i(k) + J^{\mu i}(-k)a_\mu^i(k) \right]. \quad (47)$$

Integrating out the connection we get

$$S^{(2)}[J] = \int \frac{d^4 k}{(2\pi)^4} \frac{1}{2} J^{\mu i}(-k) \frac{P_{\mu i|\nu j}^{(3,1)}}{k^2} J^{\nu j}(k), \quad (48)$$

where the usual $i\epsilon$ prescription is implied. The propagator is then

$$\langle a_{\mu i}(-k) a_{\nu j}(k) \rangle = \frac{1}{i} \frac{\delta}{\delta J^{\mu i}(-k)} \frac{1}{i} \frac{\delta}{\delta J^{\nu j}(k)} e^{iS[J]} \Big|_{J=0} = \frac{P_{\mu i|\nu j}^{(3,1)}}{i k^2} \quad (49)$$

4.4 The ghost sector

We do not need to add any ghosts for the gauge-fixing condition (32), or if we do these ghosts will not interact with any other fields. Thus, we only need to worry about the ghosts for fixing the non-Abelian $SO(3)$ gauge-symmetry. This is done as in the case of Yang-Mills gauge theory, with the only non-triviality being that the projector $P^{(3,1)}$ is inserted into the gauge-fixing condition (33). We thus get the following ghost sector Lagrangian

$$\mathcal{L}_{gh} = \frac{3}{2} \bar{c}^i D^\mu P_{\mu i|\nu j}^{(3,1)} \left(D^\nu c^j + \frac{i}{\sqrt{g^{(2)}}} \epsilon^{jkl} a^{\nu k} c^l \right), \quad (50)$$

where c^i, \bar{c}^i are ghosts, and we have extended the derivative to the full, non-linear covariant derivative including the connection perturbation. We have also multiplied the ghost Lagrangian by $3/2$ to have the standard ghost $1/ik^2$ propagator. The quantity $g^{(2)}$ is a coupling constant that in particular appears in the rescaling of the connection perturbation in order to put its kinetic term in the canonical form (46).

In the Minkowski limit, in which all computations will be done, we can replace $D_\mu \rightarrow \partial_\mu$. The Σ -part of the projector in the kinetic term can then be dropped, as the partial derivatives commute. We then end up with the standard Yang-Mills theory ghost propagator

$$\langle \bar{c}^i(-k) c^j(k) \rangle = \frac{\delta^{ij}}{ik^2}. \quad (51)$$

5 Interactions

Having derived the propagator we only need the Feynman rule vertex factors, as well as polarizations to be used to project the external legs of diagrams onto physical graviton scattering amplitudes. We have already gave expressions (16) for the later when spelling out the mode decomposition. However, we will also need the covariant versions, and these will require introducing spinors. At the same time can get sufficiently far in the analysis of interactions without using spinors. Thus, we first work out the interactions.

5.1 Decomposition of Da

Before we do the algebra that exhibits the structure of the interaction vertices, let us introduce a convenient representation for the Lie-algebra valued two-form $D_{[\mu} a_{\nu]}^i$. We can write

$$D_{[\mu} a_{\nu]}^i = \frac{1}{4} (Da)^{ij} \Sigma_{\mu\nu}^j + \frac{1}{8} \epsilon^{ijk} (D^\rho a_\rho^k) \Sigma_{\mu\nu}^j + (\widetilde{Da}^i)_{\mu\nu}, \quad (52)$$

where

$$(Da)^{ij} := \Sigma^{\mu\nu(i} D_\mu a_{\nu}^{j)}, \quad (Da)^{ij} = (Da)^{(ij)}, \quad \text{Tr}(Da) = 0 \quad (53)$$

is the symmetric tracefree matrix that encodes the self-dual components of $D_{[\mu} a_{\nu]}^i$, and $(\widetilde{Da}^i)_{\mu\nu}$ stands for the anti-self-dual part

$$(\widetilde{Da}^i)_{\mu\nu} := (Da^i)_{\mu\nu}^{\text{asd}}. \quad (54)$$

Our gauge-fixing condition (31), together with the fact that the propagator contains the $P^{(3,1)}$ projector, implies that in all vertices and on all the lines, internal and external, the connection a_μ^i can be taken to belong to just its $S_+^3 \otimes S_-$ irreducible component. Indeed, on the internal lines this projection is carried out by the propagator. On the external lines it will be performed by the polarization

tensors, see below. We can thus use the gauge-fixing condition for a_μ^i . Then the matrix $\Sigma^{\mu\nu i} D_\mu a_\nu^j$ is traceless, and we have denoted its symmetric part by $(Da)^{ij}$ and wrote the anti-symmetric part as a separate (second) term in (52). As we shall see below, the above components of $D_{[\mu} a_{\nu]}^i$ encode different information, and this is why it is convenient to separate the self- and anti-self-dual parts of $D_{[\mu} a_{\nu]}^i$ in the vertices.

Let us use the above expansion of Da to rewrite some terms that frequently appear in the interaction vertices. We have

$$\frac{1}{i} \epsilon^{\mu\nu\rho\sigma} D_\mu a_\nu^i D_\rho a_\sigma^j = \frac{1}{2} (Da)^{ik} (Da)^{kj} + \frac{1}{2} (Da)^{k(i} \epsilon^{j)kl} (D^\mu a_\mu^l) + \frac{1}{8} (\delta^{ij} \delta^{kl} - \delta^{ik} \delta^{jl}) (D^\mu a_\mu^k) (D^\nu a_\nu^l) - 2(\widetilde{Da}^i)^{\mu\nu} (\widetilde{Da}^j)_{\mu\nu} \quad (55)$$

and

$$\frac{1}{i} \epsilon^{\mu\nu\rho\sigma} D_\mu a_\nu^i \epsilon^{jkl} a_\rho^k a_\sigma^l = \frac{1}{2} (Da)^{ik} (\Sigma \epsilon aa)^{kj} + \frac{1}{4} \epsilon^{ikl} (\Sigma \epsilon aa)^{kj} (D^\mu a_\mu^l) - 2(\widetilde{Da}^i)^{\mu\nu} \epsilon^{jkl} a_\mu^k a_\nu^l, \quad (56)$$

where we have introduced

$$(\Sigma \epsilon aa)^{ij} := \Sigma^{\mu\nu i} \epsilon^{jkl} a_\mu^k a_\nu^l. \quad (57)$$

We note that $(\Sigma \epsilon aa)^{ij}$ is automatically symmetric as a consequence of (31).

5.2 Cubic interaction

The cubic interaction vertex is obtained from the third order terms in the expansion of the action (8). Dividing the third variation by $3!$, and rescaling the variation of the connection $\delta A_\mu^i = (i/\sqrt{g^{(2)}}) a_\mu^i$ we get the following third order Lagrangian

$$3i M^2 (g^{(2)})^{3/2} \mathcal{L}^{(3)} = g^{(3)} (Da)^{ij} (Da)^{jk} (Da)^{ki} - \frac{3g^{(2)}}{2} \left(\frac{1}{i} \epsilon^{\mu\nu\rho\sigma} D_\mu a_\nu^i D_\rho a_\sigma^j - M^2 (\Sigma \epsilon aa)^{ij} \right) (Da)^{ij} - \frac{1}{i} f(\delta) M^2 \epsilon^{\mu\nu\rho\sigma} D_\mu a_\nu^i \epsilon^{ijk} a_\rho^j a_\sigma^k. \quad (58)$$

We now use (55) and (56) to rewrite (58) as

$$3i M^2 (g^{(2)})^{3/2} \mathcal{L}^{(3)} = \left(g^{(3)} - \frac{3g^{(2)}}{4} \right) (Da)^{ij} (Da)^{jk} (Da)^{ki} + \frac{3g^{(2)}}{16} (Da)^{ij} (D^\mu a_\mu^i) (D^\nu a_\nu^j) + 3g^{(2)} (Da)^{ij} (\widetilde{Da}^i)^{\mu\nu} (\widetilde{Da}^j)_{\mu\nu} + \frac{M^2}{2} (3g^{(2)} - f(\delta)) (Da)^{ij} (\Sigma \epsilon aa)^{ij} + 2M^2 f(\delta) (\widetilde{Da}^i)^{\mu\nu} \epsilon^{ijk} a_\mu^j a_\nu^k. \quad (59)$$

We note that in the case of GR, see (13), the first term in the line 1 and the second term in the line 2 above are absent and we get simply

$$iM_p M \mathcal{L}_{\text{GR}}^{(3)} = (Da)^{ij} (\widetilde{Da}^i)^{\mu\nu} (\widetilde{Da}^j)_{\mu\nu} + \frac{1}{16} (Da)^{ij} (D^\mu a_\mu^i) (D^\nu a_\nu^j) + 2M^2 (\widetilde{Da}^i)^{\mu\nu} \epsilon^{ijk} a_\mu^j a_\nu^k. \quad (60)$$

Below we shall see that only the first of these 3 terms in the cubic GR Lagrangian is important for the scattering of two gravitons of opposite helicities. Note that the terms cubic in the derivatives of the connection blow up in the limit $M \rightarrow 0$, both in the general theory case and in GR. Thus, care will have to be taken when going to this limit. Note also that the cubic interaction starts with $(\partial a)^3$ terms, and thus seems to be very different from the $(\partial h)^2 h$ cubic vertex in the metric formulation. Still, we will see that in the case of (60) one is working with just a different description of the same GR interactions of gravitons.

We would like to emphasize how much simpler the cubic vertex (60) is as compared to the 13 terms one finds in the expansion of the Einstein-Hilbert Lagrangian around the Minkowski background metric, see [13], formula (A.5) of the Appendix. The cubic vertex (60) is still more complicated than the one in the case of Yang-Mills theory, but we shall see that in many cases (e.g. for purposes of computing MHV amplitudes) one effectively needs only the first term, which is of the same degree of complexity as in the Yang-Mills case. The analogy with Yang-Mills will become even more striking when we write down the spinor expression for this cubic vertex below.

5.3 Quartic interaction

We now work out the quartic term. Dividing the fourth variation of the action from (8) by $4!$ we get

$$\begin{aligned} -12M^4(g^{(2)})^2\mathcal{L}^{(4)} = & -g^{(4)}(Da)^{ij}(Da)^{jk}(Da)^{kl}(Da)^{li} + \tilde{g}^{(4)}(Da)^{ij}(Da)^{ij}(Da)^{kl}(Da)^{kl} \\ & + 6g^{(3)}P_{ijkl}(Da)^{im}(Da)^{mj}\left(\frac{1}{i}\epsilon^{\mu\nu\rho\sigma}D_\mu a_\nu^k D_\rho a_\sigma^l - M^2(\Sigma\epsilon aa)^{kl}\right) \\ & + \frac{6}{i}g^{(2)}M^2(Da)^{ij}\epsilon^{\mu\nu\rho\sigma}D_\mu a_\nu^i \epsilon^{jkl}a_\rho^k a_\sigma^l + g^{(2)}(Da)^{ij}(Da)^{ij}\left(\frac{1}{i}\epsilon^{\mu\nu\rho\sigma}D_\mu a_\nu^k D_\rho a_\sigma^k - M^2(\Sigma\epsilon aa)^{kk}\right) \\ & - \frac{3g^{(2)}}{2}P_{ijkl}\left(\frac{1}{i}\epsilon^{\mu\nu\rho\sigma}D_\mu a_\nu^i D_\rho a_\sigma^j - M^2(\Sigma\epsilon aa)^{ij}\right)\left(\frac{1}{i}\epsilon^{\mu\nu\rho\sigma}D_\mu a_\nu^k D_\rho a_\sigma^l - M^2(\Sigma\epsilon aa)^{kl}\right) \end{aligned}$$

For purposes of this paper we will only need the 4-vertex when evaluated completely on-shell. Thus, let us use the Lorentz gauge condition $D^\mu a_\mu^i = 0$. This simplifies both (55) and (56). Using these expansions, and collecting the terms we get

$$\begin{aligned} -12M^4(g^{(2)})^2\mathcal{L}^{(4)} = & \left(-g^{(4)} + 3g^{(3)} - \frac{3g^{(2)}}{8}\right)(Da)^{ij}(Da)^{jk}(Da)^{kl}(Da)^{li} \\ & + \left(\tilde{g}^{(4)} - g^{(3)} + \frac{5g^{(2)}}{8}\right)(Da)^{ij}(Da)^{ij}(Da)^{kl}(Da)^{kl} \\ & - 3(4g^{(3)} - g^{(2)})(Da)^{ik}(Da)^{kj}(\widetilde{Da}^i)^{\mu\nu}(\widetilde{Da}^j)_{\mu\nu} + (4g^{(3)} - 3g^{(2)})(Da)^{ij}(Da)^{ij}(\widetilde{Da}^k)^{\mu\nu}(\widetilde{Da}^k)_{\mu\nu} \quad (61) \\ & - \frac{3M^2}{2}(4g^{(3)} - 3g^{(2)})\left((Da)^{ik}(Da)^{kj} - \frac{1}{3}\text{Tr}((Da)^2)\delta^{ij}\right)(\Sigma\epsilon aa)^{ij} \\ & - \frac{3g^{(2)}}{2}P_{ijkl}\left(2(\widetilde{Da}^i)^{\mu\nu}(\widetilde{Da}^j)_{\mu\nu} + M^2(\Sigma\epsilon aa)^{ij}\right)\left(2(\widetilde{Da}^k)^{\mu\nu}(\widetilde{Da}^l)_{\mu\nu} + M^2(\Sigma\epsilon aa)^{kl}\right) \\ & - 12M^2g^{(2)}(Da)^{ij}(\widetilde{Da}^i)^{\mu\nu}\epsilon^{jkl}a_\mu^k a_\nu^l. \end{aligned}$$

In the case of GR many of these terms become zero and we get a much simpler (on-shell $D^\mu a_\mu^i = 0$) 4-vertex for GR:

$$\begin{aligned} 2M_p^2M^2\mathcal{L}_{\text{GR}}^{(4)} = & (Da)^{ik}(Da)^{kj}(\widetilde{Da}^i)^{\mu\nu}(\widetilde{Da}^j)_{\mu\nu} + 2M^2(Da)^{ij}(\widetilde{Da}^i)^{\mu\nu}\epsilon^{jkl}a_\mu^k a_\nu^l \quad (62) \\ & + P_{ijkl}\left((\widetilde{Da}^i)^{\mu\nu}(\widetilde{Da}^j)_{\mu\nu} + \frac{M^2}{2}(\Sigma\epsilon aa)^{ij}\right)\left((\widetilde{Da}^k)^{\mu\nu}(\widetilde{Da}^l)_{\mu\nu} + \frac{M^2}{2}(\Sigma\epsilon aa)^{kl}\right). \end{aligned}$$

This should be compared with a much more formidable expression in the case of the metric-based GR, see [13], formula (A.6). Even with the graviton field on-shell and the background metric taken to be flat, this occupies about half a page, as compared to just two lines in (62). We also note that both the GR 4-vertex as well as the general vertex (61) start with terms $(\partial a)^4$, to be compared with just two derivatives present in the metric-based vertex $(\partial h)^2 h h$. This is part of a general pattern, and in our gauge-theoretic description the order n vertex starts from $(\partial a)^n$ terms.

5.4 Ghost sector interactions

For completeness, we also spell out the ghost sector interactions, even though we will not need this in the present paper. Thus, the second, interaction term in (50), after some simple algebra involving integrating by parts, expanding the product of two ϵ tensors and using the gauge-fixing condition (32), becomes

$$\mathcal{L}_{\text{gh}}^{\text{inter}} = -\frac{i}{\sqrt{g^{(2)}}} \partial^\mu \bar{c}^i \left(\epsilon^{ijk} \eta_{\mu\nu} - \frac{1}{2} \delta^{ij} \Sigma_{\mu\nu}^k \right) a^{\nu j} c^k, \quad (63)$$

which is different from the usual Yang-Mills ghost vertex in the fact that an extra Σ -term is present. The factors in front of this interaction vertex that come from rescaling of the connection perturbation are also unusual.

6 Spinor technology and the helicity spinors

As is common to any modern derivation of the scattering amplitudes, the formalism of helicity states turns out to be extremely convenient. These are most efficiently described using spinors, or, as some literature calls them, twistors. The recent wave of interest into the spinor helicity methods originates in [9]. The method itself is, however, at least twenty years older, see e.g. [14], [15]. We start by listing some formulas involving spinors, mainly to establish the conventions. Our notations and conventions are more similar to those in the gravity literature, but we hope that the interested particle physics reader will still be able to follow.

6.1 Soldering form

The soldering form provides a map from the space of vectors to the space of rank two spinors (with two indices of opposite types). We use the conventions with a Hermitian soldering form:

$$(\theta_\mu^{AA'})^* = \theta_\mu^{AA'}.$$

The metric is obtained as a square of the soldering form:

$$\eta_{\mu\nu} = -\theta_{\mu A}^{A'} \theta_{\nu B}^{B'} \epsilon^{AB} \epsilon_{A'B'}, \quad (64)$$

where the minus sign is dictated by our desire to work with a Hermitian soldering form, while at the same time have signature $(-, +, +, +)$. We can also rewrite this formula as

$$\eta_{\mu\nu} = \theta_{\mu A'}^A \theta_{\nu A}^{A'}, \quad (65)$$

so that the minus sign disappears. The contraction that appears in this formula, i.e. unprimed indices contracting bottom left to down right, and the primed indices contracting oppositely, will be referred to as *natural* contraction. We will sometimes use index free notation and then the natural contraction will be implied.

6.2 The spinor basis

It is very convenient to introduce in each spinor space S_+, S_- a certain spinor basis. Since each space is (complex) 2-dimensional we need two basis vectors for each space. Let us denote these by

$$o_A, \iota_A \in S_+, \quad o^{A'}, \iota^{A'} \in S_-.$$

Note that we shall assume that the basis in the space of primed spinors is the complex conjugate of the basis in the space S_+ :

$$\iota^{A'} = (\iota^A)^*, \quad o^{A'} = (o^A)^*.$$

The basis vectors are pronounced as "omicron" and "iota". Since the norm of every spinor is zero, we cannot demand that each of the basis vectors is normalised. However, we can demand that the product between the two basis vectors in each space is unity. Thus, the basis vectors satisfy the following normalisation:

$$\iota^A o_A = 1, \quad \iota^{A'} o_{A'} = 1.$$

Of course, a spinor basis in each space S_+, S_- is only defined up to an $\text{SL}(2, \mathbb{C})$ rotation. Any $\text{SL}(2, \mathbb{C})$ rotated basis gives an equally good basis, and it can be seen that any two bases can be related by a (unique) $\text{SL}(2, \mathbb{C})$ rotation.

Once a spinor basis is introduced, we have the following expansion of the spinor metric, that is the ϵ_{AB} symbol

$$\epsilon_{AB} = o_A \iota_B - \iota_A o_B. \quad (66)$$

A similar formula is also valid for $\epsilon_{A'B'}$.

6.3 The soldering form in the spinor basis

The following explicit expression for the soldering form $\theta_{\mu A}^{A'}$ in terms of the basis one-forms t_μ and x_μ, y_μ, z_μ , as well as the spinor basis vectors $o^A, o^{A'}, \iota^A, \iota^{A'}$ can be obtained:

$$\theta_{\mu}^{AA'} = \frac{t_\mu}{\sqrt{2}}(o^A o^{A'} + \iota^A \iota^{A'}) + \frac{z_\mu}{\sqrt{2}}(o^A o^{A'} - \iota^A \iota^{A'}) + \frac{x_\mu}{\sqrt{2}}(o^A \iota^{A'} + \iota^A o^{A'}) + \frac{iy_\mu}{\sqrt{2}}(o^A \iota^{A'} - \iota^A o^{A'}).$$

Note that the above expression is explicitly Hermitian.

6.4 A doubly null tetrad

Collecting the components in front of equal spinor combinations in the above formula for the soldering form we can rewrite it as:

$$\theta_{\mu}^{AA'} = l_{\mu} o^A o^{A'} + n_{\mu} \iota^A \iota^{A'} + m_{\mu} o^A \iota^{A'} + \bar{m}_{\mu} \iota^A o^{A'}, \quad (67)$$

where

$$l_{\mu} = \frac{t_{\mu} + z_{\mu}}{\sqrt{2}}, \quad n_{\mu} = \frac{t_{\mu} - z_{\mu}}{\sqrt{2}}, \quad m_{\mu} = \frac{x_{\mu} + iy_{\mu}}{\sqrt{2}}, \quad \bar{m}_{\mu} = \frac{x_{\mu} - iy_{\mu}}{\sqrt{2}}.$$

Note that l, n are real one-forms, while $\bar{m}_{\mu} = m_{\mu}^*$. The above collection of one-forms is known as a *doubly null tetrad*. Indeed, it is easy to see that all 4 one-forms introduced above are null, e.g. $l^{\mu} l_{\mu} = 0$. The only non-zero products are

$$l^{\mu} n_{\mu} = -1, \quad m^{\mu} \bar{m}_{\mu} = 1.$$

Thus, the Minkowski metric can be written in terms of a doubly null tetrad as

$$\eta_{\mu\nu} = -l_{\mu} n_{\nu} - n_{\mu} l_{\nu} + m_{\mu} \bar{m}_{\nu} + \bar{m}_{\mu} m_{\nu},$$

which can also be verified directly by substituting (67) into the formula (64) for the metric.

6.5 Self-dual two-forms

The self-dual two-forms that play the central role in this article can be written down more naturally (i.e. without any reference to the time plus space decomposition of the tetrad internal index) in terms of spinors. We use the following definition:

$$\Sigma^{AB} := \frac{1}{2} \theta^A_{A'} \wedge \theta^{BA'}, \quad (68)$$

or, without the form notation

$$\Sigma_{\mu\nu}^{AB} = \theta_{\mu A'}^{(A} \theta_{\nu}^{B)A'}, \quad (69)$$

where we used the fact that symmetrization on the unprimed spinor indices has the same effect as anti-symmetrization on the spacetime indices.

Explicitly, in terms of the null tetrad and the spinor basis we get

$$\Sigma^{AB} = l \wedge m o^A o^B + \bar{m} \wedge n i^A i^B + (l \wedge n - m \wedge \bar{m}) i^{(A} o^{B)}. \quad (70)$$

Let us also give the following useful formula for the decomposition of a contraction of two soldering forms (via a primed index) in terms of the metric and the self-dual two-forms:

$$\theta_{\mu A'}^A \theta_{\nu}^{BA'} = -\frac{1}{2} \epsilon^{AB} g_{\mu\nu} + \Sigma_{\mu\nu}^{AB}. \quad (71)$$

This is easily checked by either contracting with ϵ_{AB} , which produces minus the metric on both sides, or by symmetrizing with respect to AB , which reproduces (69).

6.6 SU(2) spinors

We need to introduce the notion of SU(2) spinors when we consider the Hamiltonian formulation of any fermionic theory. In our case, we need this notion to establish a relation between our polarization tensors (16) and some spacetime covariant expressions that we shall write down below.

To define SU(2) spinors we need a Hermitian positive-definite form on spinors. This is a rank 2 mixed spinor $G_{A'A} : G_{A'A}^* = G_{A'A}$, such that for any spinor λ^A we have $(\lambda^*)^{A'} \lambda^A G_{A'A} > 0$. Here $(\lambda^*)^{A'}$ is the complex conjugate of λ^A . We can define the SU(2) transformations to be those SL(2, \mathbb{C}) ones that preserve the form $G_{A'A}$. Then $G_{A'A}$ defines an anti-linear operation \star on spinors via:

$$(\lambda^*)_A := G_{AA'} (\lambda^*)^{A'}. \quad (72)$$

We require that the anti-symmetric rank 2 spinor ϵ_{AB} is preserved by the \star -operation:

$$(\epsilon^*)_{AB} = \epsilon_{AB}, \quad (73)$$

which implies the following normalisation condition

$$G_{AA'} G^{A'}_B = \epsilon_{AB}. \quad (74)$$

Using the normalisation condition we find that $(\lambda^{\star\star})^A = -\lambda^A$ or

$$\star^2 = -1. \quad (75)$$

Thus, the \star -operation so defined is similar to a "complex structure", except for the fact that it is anti-linear:

$$(\alpha \lambda^A + \beta \eta^A)^\star = \bar{\alpha} (\lambda^*)^A + \bar{\beta} (\eta^*)^A. \quad (76)$$

Now for the purpose of comparing to results of the 3+1 decomposition, we need to introduce a special Hermitian form that arises once a time vector field is chosen. We can consider the zeroth component of the soldering form

$$\theta_0^{AA'} \equiv \theta_{\mu}^{AA'} \left(\frac{\partial}{\partial t} \right)^{\mu} = \frac{1}{\sqrt{2}} \left(o^A o^{A'} + \iota^A \iota^{A'} \right). \quad (77)$$

It is Hermitian, and so we can use a multiple of $\theta_0^{AA'}$ as $G^{AA'}$. It remains to satisfy the normalisation condition (74). This is achieved by

$$G^{AA'} := \sqrt{2}\theta_0^{AA'}. \quad (78)$$

We then define the spatial soldering form via

$$\sigma^{iAB} := G^A_{A'} \theta^{iBA'}, \quad (79)$$

which is automatically symmetric $\sigma^{iAB} = \sigma^{i(BA)}$ because its anti-symmetric part is proportional to the product of the time vector with a spatial vector, which is zero. Explicitly, in terms of the spinor basis introduced above we have

$$\sigma^{iAB} = m^i o_A o_B - \bar{m}^i \iota_A \iota_B - \frac{z^i}{\sqrt{2}} (\iota_A o_B + o_A \iota_B). \quad (80)$$

The action of the \star -operation on the basis spinors is as follows:

$$(o^\star)^A = -\iota^A, \quad (\iota^\star)^A = o^A. \quad (81)$$

It is then easy to see from (80) that the spatial soldering form so defined is anti-Hermitian with respect to the \star operation:

$$(\sigma^{i\star})^{AB} = -\sigma^{iAB}. \quad (82)$$

It is not hard to deduce the following property of the product of two spatial soldering forms:

$$\sigma_A^{iB} \sigma_B^{jC} = \frac{1}{2} \delta^{ij} \epsilon_A^C + \frac{i}{\sqrt{2}} \epsilon^{ijk} \sigma_A^k{}^C. \quad (83)$$

6.7 Converting to the spinor form

We are now ready to use the spinor objects introduced above. First, let us discuss how the expressions written in $\text{SO}(3)$ notations used so far can be converted into spinor notations. Indeed, we have so far worked with the connection perturbation being a_μ^i . It is now convenient to pass to the spinor description, in which all indices of a_μ^i are converted into spinor ones. This is done with the soldering form for the spacetime index, and with Pauli matrices for the internal one.

To fix the form of the multiple of the Pauli matrices that is relevant here, we will require that under this map the identity matrix δ^{ij} becomes the matrix $\epsilon^{(A|C|} \epsilon^{B)D}$ in spinor notation. Indeed, both matrices have trace 3. Thus, we denote the map from objects with $\text{SO}(3)$ indices to those with pairs of unprimed spinor indices by T^{iAB} and require it to have the property:

$$\delta^{ij} T^{iAB} T^{jCD} = \epsilon^{(A|C|} \epsilon^{B)D}. \quad (84)$$

This fixes T^{iAB} up to a sign.

Now, to determine what multiple of this object appears in the relation between $\Sigma_{\mu\nu}^i$ and $\Sigma_{\mu\nu}^{AB}$, both of which have been defined before, we need to look into the algebra satisfied by them. We have the algebra used many times in preceding text:

$$\Sigma_\mu^i{}^\rho \Sigma_{\rho\nu}^j = -\delta^{ij} \eta_{\mu\nu} + \epsilon^{ijk} \Sigma_{\mu\nu}^k. \quad (85)$$

At the same time, a simple computation of the same contraction of $\Sigma_{\mu\nu}^{AB}$ gives

$$\Sigma_\mu^{AB\rho} \Sigma_{\rho\nu}^{CD} = -\frac{1}{2} \epsilon^{(A|C|} \epsilon^{B)D} \eta_{\mu\nu} + \frac{1}{2} \epsilon^{A(C} \Sigma_{\mu\nu}^{D)B} + \frac{1}{2} \epsilon^{B(C} \Sigma_{\mu\nu}^{D)A}. \quad (86)$$

The coefficient in front of the first term here is half that in (85). We thus learn that there is a factor of $\sqrt{2}$ in the conversion of an $\text{SO}(3)$ index into a pair of spinor ones:

$$\frac{1}{\sqrt{2}} \Sigma_{\mu\nu}^i T^{iAB} = \Sigma_{\mu\nu}^{AB}. \quad (87)$$

We can also immediately write down the conversion rule of the ϵ^{ijk} tensor by comparing the second terms in (85) and (86). We get

$$\sqrt{2} \epsilon^{(AB)(CD)}{}_{(EF)} X^{(EF)} = \epsilon^{A(C} X^{D)B} + \epsilon^{B(C} X^{D)A}. \quad (88)$$

Overall, we find that the matrices T^{iAB} satisfy the following algebra:

$$T^{iAB} T_B^{jC} = -\frac{1}{2} \delta^{ij} \epsilon^{AC} - \frac{1}{\sqrt{2}} \epsilon^{ijk} T^{kAC}, \quad (89)$$

which fixes them uniquely. We see that these quantities are just

$$T^{iAB} = i\sigma^{iAB},$$

where σ^{iAB} are the spatial soldering forms introduced above. Explicitly, in terms of the spinor basis, as well as a basis m^i, \bar{m}^i, z^i in \mathbb{R}^3 we have:

$$T^{iAB} = i m^i o_A o_B - i \bar{m}^i \iota_A \iota_B - \frac{i}{\sqrt{2}} z^i (\iota_A o_B + o_A \iota_B). \quad (90)$$

Note that T^{iAB} is \star -Hermitian, i.e. $(T^{i\star})^{AB} = T^{iAB}$.

6.8 Further on spinor conversion

Let us now discuss the rules of dealing with the spacetime indices. Each such index has to be converted into a mixed type pair of spinor indices using the soldering form $\theta^{\mu AA'}$. We shall refer to the operator of the partial derivative with its spacetime index converted into a pair of spinor indices as the Dirac operator:

$$\partial_\mu := \theta_{\mu AA'} \partial^{AA'}. \quad (91)$$

Note that, because of our signature choice, and thus a minus sign in (64), we have $\partial^{AA'} = -\theta^{\mu AA'} \partial_\mu$. One has to be careful about these minus signs.

We now come to objects that have both types of indices, spacetime and internal. The conversion of these is that we write them as the corresponding soldering forms times objects with only spinor indices. Thus, e.g. for the connection we write

$$a_\mu^i = T^{iAB} \theta_\mu^{MM'} a_{ABMM'}, \quad (92)$$

which defines what we mean by the connection with all its indices translated into the spinor ones. This choice of the normalization factor in the above formula is convenient, because as we already discussed before the Kronecker delta δ^{ij} goes under this map into the object $\epsilon^{(A} \epsilon^{B)}{}_{D}$, which is the identity map on the space of symmetric rank two spinors. The only unusual translation rule is

$$\Sigma_{\mu\nu}^i := \sqrt{2} T^{iAB} \theta_\mu^{MM'} \theta_\nu^{NN'} \Sigma_{MM'NN'}^{AB}, \quad (93)$$

where

$$\Sigma_{MM'NN'}^{AB} = \epsilon^{(A}{}_{M} \epsilon^{B)}{}_{N} \epsilon_{M'N'}. \quad (94)$$

The only reason for putting the factor of $\sqrt{2}$ in (93) is that the objects so defined have the same algebra (85) as we are used to, and as was used on multiple occasions in deriving the form of the interaction terms in the Lagrangian.

The final useful formula for the purposes of conversion is

$$\theta_{[\mu}^{MM'} \theta_{\nu]}^{NN'} = \frac{1}{2} \epsilon^{NM} \Sigma_{\mu\nu}^{M'N'} + \frac{1}{2} \epsilon^{M'N'} \Sigma_{\mu\nu}^{MN}, \quad (95)$$

where

$$\Sigma_{\mu\nu}^{MN} = \theta_{\mu M'}^{(M} \theta_{\nu}^{N)M'}, \quad \Sigma^{M'N'} = \theta_{\mu}^{M(M'} \theta_{\nu M}^{N')}. \quad (96)$$

Note that the natural contractions appear in these definitions. The formula (95) then implies

$$V_{[\mu} U_{\nu]} = \frac{1}{2} V_{MM'} U_N^{M'} \Sigma_{\mu\nu}^{MN} + \frac{1}{2} V^M{}_{M'} U_{MN'} \Sigma_{\mu\nu}^{M'N'}, \quad (97)$$

where again natural contractions appear. The first term here is the self-dual part, and the second is anti-self-dual part of the two-form $V_{[\mu} U_{\nu]}$.

6.9 Momentum spinors

Consider a massless particle of a particular 3-momentum vector \vec{k} . The 4-vector $k^\mu = (|k|, \vec{k})$ is then null. As such, it can be written as a product of two spinors $k^A k^{A'} = \theta_{AA'}^\mu k_\mu$. In the case of Lorentzian signature the spinors $k^A, k^{A'}$ must be complex conjugates of each other (so that the resulting null 4-vector is real). It is then clear that k^A is only defined modulo a phase. Moreover, as the vector \vec{k} varies, i.e. as $\vec{n} = \vec{k}/|k|$ varies over the sphere S^2 , there is no continuous choice of the spinor k^A . We make the following choice:

$$k^A \equiv k^A(\vec{k}) := 2^{1/4} \sqrt{\omega_k} \left(\sin(\theta/2) e^{-i\phi/2} \iota^A + \cos(\theta/2) e^{i\phi/2} o^A \right), \quad (98)$$

where o^A, ι^A is a basis in the space of unprimed spinors, and $\omega_k = |k|$. Here θ, ϕ are the usual coordinates on S^2 so that the momentum vector in the direction of the positive z-axes corresponds to $\theta = \phi = 0$. We see that the corresponding spinor is $2^{1/4} \sqrt{\omega_k} o^A$. The formula (98) can be checked using the expression (67) for the soldering form.

We can now see effects of the change of the momentum vector direction. Consider, for example, what happens when the momentum direction gets reversed. This corresponds to $\theta \rightarrow \pi - \theta$ and $\phi \rightarrow \phi + \pi$. We get

$$k^A(-\vec{k}) = i 2^{1/4} \sqrt{\omega_k} \left(-\cos(\theta/2) e^{-i\phi/2} \iota^A + \sin(\theta/2) e^{i\phi/2} o^A \right). \quad (99)$$

We now note that

$$k^A(-k) = i(k^\star)^A(k), \quad (100)$$

where the action of the \star -operation on the basis spinors is given in (81). Now, using the fact that $\star^2 = -1$ it is easy to see that flipping the sign of the momentum twice we get minus the original momentum spinor. In other words, k^A takes values in a non-trivial spinor bundle over S^2 .

6.10 Helicity spinors

The aim of this subsection is to use the rules for the $i \rightarrow (AB)$ conversion deduced above, as well as the definition (98) of the momentum spinors k^A to write down convenient expressions for the polarization tensors (16) in the spinor language.

Our polarization tensors are built from the vectors $m^i(k)$, $\bar{m}^i(k)$, where the direction of the z^i axes is chosen to be that of the momentum 3-vector \vec{k} . Thus, let us start by assuming that \vec{k} points along the positive z -direction. Then from (90) we have $T^{iAB}m^i = -i\iota^A\iota^B$, $T^{iAB}\bar{m}^i = i o^A o^B$, and therefore

$$m^i m^j \rightarrow -\iota^A \iota^B \iota^C \iota^D, \quad \bar{m}^i \bar{m}^j \rightarrow -o^A o^B o^C o^D \quad (101)$$

when converted to spinor notations. We can, however, use the available freedom of gauge $SO(3)$ rotations and consider polarization tensors (spinors) more general than those above. Indeed, we can always shift our (spatial projection of the) connection by a gauge transformation $a_{ij} \rightarrow a_{ij} + (\partial_{(i}\phi_{j)})^{tf}$, where also the tracefree part needs to be taken in order to preserve the tracelessness of the a_{ij} . Such a shift being pure gauge, it does not have any effect on the scattering amplitudes. So, we can freely add to both polarization tensors an object of the form $z_{(i}\phi_{j)}$, where again a tracefree part is assumed. Moreover, the vector ϕ_i can be different for the positive and negative helicity polarizations. Using the spinor conversion rules written above, it is not hard to see that this means that one will obtain correct scattering amplitudes when using instead of (101) the following expressions

$$\iota^A \iota^B \iota^C \iota^D \rightarrow (\iota + \alpha o)^A (\iota + \beta o)^B (\iota + \gamma o)^C \iota^D, \quad o^A o^B o^C o^D \rightarrow (o + \alpha' \iota)^A (o + \beta' \iota)^B (o + \gamma' \iota)^C o^D,$$

for arbitrary coefficients $\alpha, \beta, \gamma, \alpha', \beta', \gamma'$. For reasons to become clear below, the most convenient choice is

$$\iota^A \iota^B \iota^C \iota^D \rightarrow \frac{q^{(A} q^B q^C \iota^{D)}}{(q^E o_E)^3}, \quad o^A o^B o^C o^D \rightarrow \frac{o^{(A} o^B o^C (p^*)^D)}{(\iota^E (p^*)_E)}, \quad (102)$$

where q^A, p^A are arbitrary spinors, and \star is the operation on $SU(2)$ spinors introduced above. Note that while in the choice of the first polarization spinor we have replaced as many as 3 copies of ι^A by an arbitrary reference spinor q^A , in the second polarization we only changed a single copy of o^A to $(p^*)^A$. The reason for this will become clear below.

Let us now rewrite the spinor expressions for the full polarization tensors (16) using the spinors $k^A = 2^{1/4} \sqrt{\omega_k} o^A$ and $(k^*)^A = -2^{1/4} \sqrt{\omega_k} \iota^A$. We get

$$\varepsilon^{-ABCD}(k) = M \frac{q^{(A} q^B q^C (k^*)^D)}{(q^E k_E)^3}, \quad \varepsilon^{+ABCD}(k) = \frac{1}{M} \frac{k^{(A} k^B k^C (p^*)^D)}{(k^*)^E (p^*)_E}. \quad (103)$$

Note that all the annoying factors of $\sqrt{2}$ in the original formulas (16), as well as some minus signs present in the intermediate expressions, have now cancelled. Note also that while the previous spinor expressions were only valid in a frame where the 3-momentum was pointing in the z -direction, the expressions (103) are valid in an arbitrary frame.

It remains to observe that one will obtain (103) as the spin 2 parts of the spatial projections of the following mixed spinors:

$$\varepsilon^{-ABCA'}(k) = M \frac{q^A q^B q^C k^{A'}}{\langle q k \rangle^3}, \quad \varepsilon^{+ABCA'}(k) = \frac{1}{M} \frac{k^A k^B k^C p^{A'}}{[p k]}, \quad (104)$$

where we have introduced the usual notations for the spinor contractions

$$\langle \lambda \eta \rangle := \lambda^A \eta_A, \quad [\lambda \eta] = \lambda_{A'} \eta^{A'}. \quad (105)$$

We note that the helicity spinors are normalized so that

$$\varepsilon^{-ABCA'} \epsilon_{ABCA'}^+ = 1. \quad (106)$$

The expressions (104) are the main outcome of this heavy in conventions section. We note that these expressions could have been guessed as the only ones with the correct dimensions, as well as with

the right homogeneity degree zero dependence on the reference spinors $q^A, p^{A'}$, and the right degree of homogeneity under the rescaling of the momentum spinors $k^A \rightarrow tk^A, k^{A'} \rightarrow t^{-1}k^{A'}$. Indeed, it is clear that under these rescalings (keeping the 4-momentum $k^A k^{A'}$ unchanged) we get

$$\varepsilon^{-ABCA'}(k) \rightarrow t^{-4}\varepsilon^{-ABCA'}(k), \quad \varepsilon^{+ABCA'}(k) \rightarrow t^4\varepsilon^{+ABCA'}(k). \quad (107)$$

However, under any such a guess possibly important numerical factors could have been missed, and it is gratifying to see that after establishing all the conversion formulas, the helicity spinors turned out to be just the simplest expressions possible, without any complicating numerical prefactors. We note that the final spacetime covariant expressions (104) explain our choice (102) at the level of the spatially projected expressions.

The only complication that remains to be discussed is the fact that the positive helicity gravitons have to be taken to be slightly massive, as we have seen in the section on the Minkowski space limit. Because of this, the meaning of the spinor k^A that is used in the positive helicity spinor in (104) is not yet defined. To settle this, we shall represent the massive 4-vector $k^2 = 2M^2$ of the positive helicity gravitons as follows

$$k^{AA'} = k^A k^{A'} + M^2 \frac{p^A p^{A'}}{\langle p k \rangle [p k]}. \quad (108)$$

This gives precisely the required $k^2 = -k^{AA'} k_{AA'} = 2M^2$. Here $p^A p^{A'}$ are a reference spinor and its complex conjugate. At this point it can be arbitrary, but it is convenient to take it to be the same as the one that appears in the positive helicity spinor in (104). It is now the spinor k^A that appears in the decomposition (108) is what one has to use in the positive helicity spinor in (104). We emphasize that only the positive helicity momentum 4-vectors should be taken to be massive, while the negative helicity does not need this complications, and the corresponding momenta 4-vectors come directly as a product of two spinors.

6.11 A relation to the metric helicity states

It is instructive to see how the metric description helicity spinors can be obtained from the expressions (104). For this we need to recall the passage to the metric perturbation variable that was explained in great detail in [8]. In that paper we have seen that the metric perturbation is obtained by applying to the connection the operator \bar{D} . One should also rescale by $1/M$ to keep the mass dimension correct. At the spacetime covariant level the operator \bar{D} corresponds to the operation of taking the anti-self-dual part of the two-form da^i . In the spinor notations, this boils down to the following expression for the metric perturbation

$$h_{AB A' B'} \sim \frac{1}{M} \partial_{A'}^E a_{B' E A B}, \quad (109)$$

where $\partial_{AA'} = -\theta_{AA'}^\mu \partial_\mu$ is the Dirac operator, and the \sim sign means that we are only interested in this relation modulo numerical factors. Applying this to the helicity spinors (104), and ignoring the arising numerical factors, one immediately sees that the usual metric spinor helicity states get reproduced:

$$h_{AA' BB'}^-(k) \sim \frac{q_A k_{A'} q_B k_{B'}}{\langle q k \rangle^2}, \quad h_{AA' BB'}^+(k) \sim \frac{k_A p_{A'} k_B p_{B'}}{[p k]^2}. \quad (110)$$

Note that for the negative helicity the metric formulation helicity spinor arises by a single q spinor in the numerator of (104) contracting with the momentum k spinor, removing one of the factors of $\langle q k \rangle$ from the denominator. There is also the cancellation of the factor of M in the connection helicity spinor with $1/M$ in the passage to the metric perturbation. For the positive helicity the mechanism of obtaining the usual metric formulation helicity spinor is more subtle. Indeed, if the positive helicity

graviton 4-momentum was null, then we would be contracting two momentum k spinors, which would give a zero result. Instead, it is the second, mass term in (108) that gives a non-zero contribution. The factor of M^2 in this second term then is nicely cancelled by the $1/M$ in the helicity spinor and the additional factor of $1/M$ in (109). We therefore see that it is essential that the positive helicity graviton is kept massive till the Minkowski limit can be taken.

Once again, the fact that the usual metric helicity states get reproduced could be taken as the sufficient reason to work with (104). However, we find the given above derivation of (104) that does not involve any reference to the metric more self-contained.

7 Feynman rules in the spinor form

Now that we understand how expressions can be converted into the spinor language, we can write down the derived above Feynman rules in the spinor form. As we shall see, there are many advantages in working with spinors, as some operations that are not easy to deal with in the $SO(3)$ notation become elementary once one expresses them using spinors. The prime example is the projection on the $S_+^3 \otimes S_-$ representation of the Lorentz group that appeared in our derivation of the propagator. In the spinor language this simply corresponds to the symmetrization on the unprimed spinor indices. We shall also see that the interaction vertices simplify considerably.

7.1 Propagator

We have previously found the propagator to be given by $1/ik^2$ times the projector $P_{\mu i|\nu j}^{(3,1)}$, given in (30), on the $S_+^3 \otimes S_-$ irreducible component of objects of type a_μ^i with one spacetime and one internal index. To find what this projector becomes once converted into the spinor form one can multiply the spacetime indices with the soldering forms, and the internal indices with T^{iAB} . However, one does not need to do this computation as it is clear that this projector is simply the product of the identity operator acting on the primed index, times the operator of symmetrization of the 3 unprimed spinor indices. Thus, we can write

$$\langle a_{EFG E'}(-k) a^{ABCA'}(k) \rangle = \frac{1}{ik^2} \epsilon_E^{(A} \epsilon_F^B \epsilon_G^{C)} \epsilon_{E'}^{A'}. \quad (111)$$

7.2 Pieces of the interaction vertices

Here we develop a dictionary translating the various blocks that appear in the interaction vertices into the spinor form. As we recall from (52), one of the main building blocks of the vertices is the two-form $D_{[\mu} a_{\nu]}^i$, and various quantities constructed from it. We remind that from now on we replace the covariant derivative D by the partial one.

The first block to be translated is the $(Da)^{ij}$ symmetric tracefree matrix, whose definition is given in (53). Applying the rules given above we get:

$$(Da)^{ij} \rightarrow \sqrt{2} \partial^{(A} a^{BCD)M'}_{M'}, \quad (112)$$

where the result is easy to understand, and the factor of $\sqrt{2}$ comes from the same factor in the translation (93) of $\Sigma_{\mu\nu}^i$. Further, we have

$$D^\mu a_\mu^i \rightarrow \partial_{M'}^M a^{AB}_{M'}{}^{M'}. \quad (113)$$

Finally, we have

$$(\widetilde{Da}^i)_{\mu\nu} \rightarrow \frac{1}{2} \Sigma_{\mu\nu}^{M'N'} \partial^M a^{AB}_{M'}{}^{M'}. \quad (114)$$

We also need the spinor representation of the two-form $\epsilon^{ijk}a_\mu^j a_\nu^k$. Its self-dual part encoded in the matrix (57) is given by

$$(\Sigma\epsilon aa)^{ij} \rightarrow -a^{CE(A}_{M'}a^B)_{E}{}^{DM'} - a^{DE(A}_{M'}a^B)_{E}{}^{CM'}. \quad (115)$$

It is $(AB) \rightarrow (CD)$ symmetric, but it has trace given by the full contraction of the two connections. The anti-self-dual part of $\epsilon^{ijk}a_\mu^j a_\nu^k$ is given by

$$(\epsilon^{ijk}a_\mu^j a_\nu^k)_{asd} \rightarrow -\frac{1}{\sqrt{2}}\Sigma_{\mu\nu}^{M'N'} a^{EF(A}_{M'}a^B)_{EFN'}. \quad (116)$$

We now introduce some notations to simplify the above spinorial expressions. The idea behind this notation is that we omit pairs of naturally contracted indices. Thus, we define

$$\begin{aligned} (\partial a)^{ABCD} &:= \partial^{(A}_{M'}a^{BCD)M'}, & (\partial a)^{M'N'AB} &:= \partial^{C(M'}a_C^{ABN')}, & (\partial a)^{AB} &:= \partial_{M'}^M a^{AB}{}_M{}^{M'}, \\ (aa)^{ABCD} &:= a^{ABM}_{M'}a^{CD}{}_M{}^{M'}, & (aa)^{M'N'CD} &:= a^{CD(AM'}a_{CD}{}^{B)N'}. \end{aligned} \quad (117)$$

The blocks appearing in the interaction vertices can then be written very compactly in terms of these quantities.

7.3 Cubic interaction

For computation of Minkowski limit scattering amplitudes we should only keep the terms with the highest power of the derivative at each order of the perturbation theory. Indeed, we have already neglected terms of the order of M as compared with ω_k by replacing the covariant derivative with the ordinary one. For consistency, we should also neglect the terms $M^2 a$ as compared to $(\partial a)^2$. This allows us to neglect the terms proportional to M^2 in (59). We can thus write

$$\begin{aligned} 3iM^2(g^{(2)})^{3/2}\mathcal{L}^{(3)} &= 2^{3/2}\left(g^{(3)} - \frac{3g^{(2)}}{4}\right)(\partial a)_{AB}{}^{CD}(\partial a)_{CD}{}^{EF}(\partial a)_{EF}{}^{AB} \\ &+ 2^{-1/2}3g^{(2)}(\partial a)^{ABCD}(\partial a)_{M'N'}{}^{AB}(\partial a)^{M'N'}{}_{CD} + 2^{1/2}\frac{3g^{(2)}}{16}(\partial a)^{ABCD}(\partial a)_{AB}(\partial a)_{CD}, \end{aligned} \quad (118)$$

where it is understood that only the Minkowski limit relevant terms are kept, and an additional factor of $1/2$ in the first term in the second line came from the factors of $1/2$ in (114), with one of them being cancelled by the factor of 2 that appears in the contraction of two Σ 's.

The last term in (118) is only (possibly) relevant for loop computations, for in any tree diagram at least one of the two factors of $(\partial a)^{AB}$ gets hit by an external state, which gives zero. Thus, we shall ignore this term in the present paper. Let us now consider the second term in (118) in more details, which is also the only term remaining in the GR cubic interaction.

7.4 The parity-preserving cubic vertex

In the case of GR the coefficient in front of the first term in (118) becomes zero, and we are left with the simple

$$i\mathcal{L}_{\text{GR}}^{(3)} = \frac{1}{2}\frac{\kappa}{M}(\partial a)^{ABCD}(\partial a)_{M'N'}{}^{AB}(\partial a)^{M'N'}{}_{CD}, \quad (119)$$

where we wrote the prefactor in a way that exhibits the symmetry factor for the last two occurrences of (∂a) . We also introduced the usual notation

$$\kappa^2 := 32\pi G, \quad \kappa = \frac{\sqrt{2}}{M_p}. \quad (120)$$

We would now like to make a pause and emphasize the similarity of (119) to the vertex of Yang-Mills theory, when rewritten in the spinor notations. Thus, one starts with the Yang-Mills Lagrangian in the form $\mathcal{L}_{\text{YM}} \sim (F_{sd})^2$, where F_{sd} is the self-dual part of the curvature. Applying the described above spinor conversion rules, at the linearized level this gives

$$\mathcal{L}_{\text{YM}}^{(2)} \sim (\partial_{A'}^{(A} A^{B)A'})^2, \quad (121)$$

where $A_{AA'}$ is the connection, and we omitted the Lie-algebra indices. Our linearized Lagrangian (14), when converted into the spinor form, is precisely analogous, except that the connection field in the case of gravity has two more unprimed spinor indices. Let us now look at the Yang-Mills cubic interaction. Again translating into the spinor form we get

$$\mathcal{L}_{\text{YM}}^{(3)} \sim (\partial_{A'}^{(A} A^{B)A'}) A_{AB'} A_B{}^{B'}. \quad (122)$$

Again, the analogy to (119) is striking. Basically, the gravitational interaction (119) is the only possible one that can be constructed following the Yang-Mills pattern, but now with more unprimed indices on the connection. Indeed, generalizing the first block $(\partial_{A'}^{(A} A^{B)A'})$ to the case with more indices one gets $(\partial_{A'}^{(A} a^{BCD)A'}) \equiv (\partial a)^{ABCD}$. We would now like to have a symmetric block involving two connections with free indices being 4 unprimed spinor indices $ABCD$. Thus, two indices must come from one connection, and the other two from the other. We also want to have some quantities constructed out of the connections that contract only in primed indices. This means that we have to convert one of the unprimed indices in each connection to a primed using the Dirac operator. This results precisely in (119). The other possible choice of having connections contracting directly, with no Dirac operators involved, i.e. $a_{AB}{}^E{}_{A'} a_{CDE}{}^{A'}$, can be easily seen not to give the desired on-shell amplitudes, see below. We thus learn that (119) is the only possible generalization of the Yang-Mills cubic vertex that gives the correct on-shell amplitudes.

Let us now check that the vertex (119), when evaluated on the graviton helicity spinors gives just the required $--+$ and $++-$ amplitudes, i.e. the squares of these amplitudes for spin one particles. Indeed, let us compute the $--+$ amplitude. The required helicity states are given in (104). We note that the combination $(\partial a)^{ABCD}$ only gives a non-zero result when applied to a positive helicity state. Thus, we must insert the positive helicity wave-function in this leg. The other two legs are symmetric, and we insert into them the remaining two negative helicity states. Let us denote the negative helicity momenta by k_1, k_2 , and the positive momentum by k_3 . After applying all the derivatives present in the vertex to their corresponding states we obtain the following contraction

$$\frac{\kappa}{M} \frac{1}{M} 3^A 3^B 3^C 3^D M \frac{q_A q_B 1_{M'} 1_{N'}}{\langle 1 q \rangle^2} M \frac{q_C q_D 2^{M'} 2^{N'}}{\langle 2 q \rangle^2} = \kappa [1 2]^2 \frac{\langle 3 q \rangle^4}{\langle 1 q \rangle^2 \langle 2 q \rangle^2}, \quad (123)$$

where we have used the usual notation $k_1^A \equiv 1^A$, etc. We can now use the momentum conservation equation, which we contract with the reference spinor q_A to get $\langle 1 q \rangle 1^{A'} + \langle 2 q \rangle 2^{A'} + \langle 3 q \rangle 3^{A'} = 0$. This immediately gives $\langle 3 q \rangle^2 / \langle 1 q \rangle^2 = [1 2]^2 / [3 2]^2$, $\langle 3 q \rangle^2 / \langle 2 q \rangle^2 = [1 2]^2 / [3 1]^2$, which allows us to rewrite (123) as

$$\mathcal{M}^{--+} = \kappa \frac{[1 2]^6}{[1 3]^2 [2 3]^2}. \quad (124)$$

This is just the expected square of the spin one result. We are ignoring some (convention dependent) factors of i that also appear, but are irrelevant for us here.

The opposite helicity configuration is computed similarly. One first notes that if two positive helicity gravitons hit the two symmetric legs of the vertex, the reference spinors $p^{A'}$ will contract, and this gives zero. Thus, the negative helicity graviton must hit one of the two symmetric legs, with one

of the positive helicity gravitons thus being necessarily inserted into the leg $(\partial a)^{ABCD}$. If we choose this to be the graviton number two, we get the following contraction

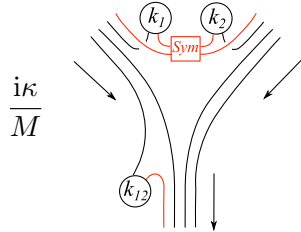
$$\frac{\kappa}{M} \frac{1}{M} 2^A 2^B 2^C 2^D M \frac{1_A 1_B p_{M'} p_{N'}}{[1p]^2} M \frac{q_C q_D 3^{M'} 3^{N'}}{\langle 3q \rangle^2} = \kappa \langle 12 \rangle^2 \frac{\langle 2q \rangle^2 [3p]^2}{\langle 3q \rangle^2 [1p]^2}. \quad (125)$$

Using the momentum conservation this becomes the required

$$\mathcal{M}^{++-} = \kappa \frac{\langle 12 \rangle^6}{\langle 13 \rangle^2 \langle 23 \rangle^2}. \quad (126)$$

We thus see that the GR vertex is parity-invariant, as it should be of course, in the sense that the amplitude for an opposite configuration of helicities is given by the complex conjugates of the original amplitude.

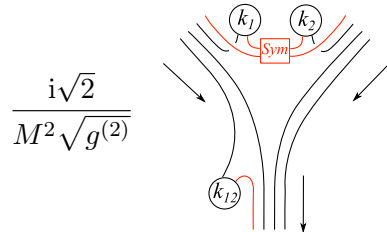
Let us now discuss the corresponding Feynman rules. When we write the vertex factor corresponding to (119) we obtain 3 terms, since the first instance of (∂a) corresponding to the self-dual part of the two-form da can be applied to any of the 3 legs of the vertex. We will not write the vertex factor as we will never need it in this paper, and because its explicit form containing many ϵ symbols and symmetrizations is not very illuminating. Instead, we shall draw a picture of the contractions involved in the vertex for just one of the terms, when momenta $k_{1,2}$ that we assume are incoming are on the lines corresponding to $(\partial a)_{asd}$, and the momentum $k_{12} = k_1 + k_2$ is on the line corresponding to $(\partial a)_{sd}$. In this picture the red lines correspond to primed, and black to unprimed spinor indices. The symbols of momenta in circles stand for the factors of $k^{AA'}$. This part of the full vertex is given by



$$\frac{i\kappa}{M} \quad (127)$$

where one also has to symmetrize over the two external legs that can get contracted to k_{12} . As we shall see later, for the purpose of computing the most interesting (graviton-graviton or more generally MHV amplitudes) this will be the only surviving contribution to the full vertex.

Let us now briefly discuss the case of a general theory. In this case the vertex of interest in the tree level computations has two pieces. The parity-invariant part is essentially the GR vertex, but with the different prefactor



$$\frac{i\sqrt{2}}{M^2 \sqrt{g^{(2)}}} \quad (128)$$

We can therefore expect that the Newton constant measuring the strength of interactions of gravitons in the general theory is

$$\frac{1}{16\pi G} = M^2 g^{(2)}, \quad (129)$$

which is essentially the coupling constant $g^{(2)}$, expressed in the units of M , the only dimensionful parameter present in the theory. This expectation will be confirmed below when we compute the parity-preserving graviton scattering amplitude.

7.5 The parity-violating cubic vertex

For a general theory there is another vertex that comes from the first term in (118). Assuming again the convention that the two momenta are incoming and one is outgoing, and taking into account the symmetry factor of $3!$, the vertex can be graphically represented as

$$\frac{i\sqrt{2}(4g^{(3)} - 3g^{(2)})}{M^2(g^{(2)})^{3/2}} \quad \begin{array}{c} \text{Diagram: A vertex with three external lines. Two lines enter from the top left and top right, labeled with momenta k_1 and k_2 respectively. One line exits from the bottom, labeled with momentum k_{12}. Each line has a box labeled 'Sym' indicating symmetry. The lines are represented by wavy lines for the incoming ones and a straight line for the outgoing one. Arrows indicate the flow of momentum: incoming for k_1 and k_2, outgoing for k_{12}. \end{array} \quad (130)$$

To see what kind of interaction this generates, let us evaluate this vertex on the graviton polarization spinors. It is immediately clear that it only produces a non-zero result when all the helicities are positive, because the negative helicity inserted into the combination $(\partial a)^{ABCD}$ gives a zero result. After applying all the derivatives to the external states, we get for the amplitude coming from this vertex

$$\mathcal{M}^{+++} = \frac{\sqrt{2}(4g^{(3)} - 3g^{(2)})}{M^5(g^{(2)})^{3/2}} \langle 12 \rangle^2 \langle 23 \rangle^2 \langle 31 \rangle^2. \quad (131)$$

Using the already known to us fact (129) that the Planck mass $M_p^2 = 1/16\pi G$ equals $M^2 g^{(2)}$ we can express the quantity $g^{(2)}$ here in terms of the Planck mass. We get

$$\mathcal{M}^{+++} = \frac{\sqrt{2}(4g^{(3)} - 3g^{(2)})}{M^2 M_p^3} \langle 12 \rangle^2 \langle 23 \rangle^2 \langle 31 \rangle^2. \quad (132)$$

It may look that this amplitude blows up as $M \rightarrow 0$. However, we first remember that it is actually zero when the momentum conservation holds and all the momenta are real. Independently of this, the question of dependence on M depends crucially on how fast the difference $4g^{(3)} - 3g^{(2)}$ goes to zero as $M \rightarrow 0$. We shall return to these questions in more details below.

We note that the amplitude of the type (132) can arise in a gravity theory with the $(Riemann)^3$ term in the Lagrangian, see e.g. [18], discussion following formula (57). However, a theory with this term in the Lagrangian would be parity-preserving, and thus would also have a non-zero \mathcal{M}^{---} amplitude. We stress that in our theories this is not the case, with only one chiral half of this amplitude being present, and the amplitude \mathcal{M}^{---} continuing to vanish as in GR. We thus learn that the higher order in curvature correction that is present in a general member of our family of theories is not $(Riemann)^3$ but rather $(Weyl_{sd})^3$, in other words the cube of the self-dual part of Weyl curvature. No anti-self-dual part cubed is present, and this is why the 3 negative helicity amplitude continues to be zero. Below we shall discuss implications of this fact in more details.

7.6 Quartic interaction

As for the cubic vertex, we now take the full expression (61), neglect the lower derivative terms that are subleading in the M/ω_k expansion, and convert everything into spinors. We get

$$\begin{aligned} -4!M^4(g^{(2)})^2\mathcal{L}^{(4)} = & \left(-8g^{(4)} + 24g^{(3)} - 3g^{(2)}\right) (\partial a)_{AB}{}^{CD}(\partial a)_{CD}{}^{EF}(\partial a)_{EF}{}^{MN}(\partial a)_{MN}{}^{AB} \\ & + \left(8\tilde{g}^{(4)} - 8g^{(3)} + 5g^{(2)}\right) (\partial a)_{AB}{}^{CD}(\partial a)_{CD}{}^{AB}(\partial a)_{EF}{}^{MN}(\partial a)_{MN}{}^{EF} \\ & - 24(4g^{(3)} - g^{(2)}) (\partial a)^{ABEF}(\partial a)_{EF}{}^{CD}(\partial a)_{M'N'}{}^{AB}(\partial a)^{M'N'}{}_{CD} \\ & + 8(4g^{(3)} - 3g^{(2)}) (\partial a)^{ABCD}(\partial a)_{ABCD}(\partial a)_{M'N'}{}^{EF}(\partial a)^{M'N'}{}_{EF} \\ & - 12g^{(2)}(\partial a)_{M'N'}{}^{(AB}(\partial a)^{CD)M'N'}(\partial a)_{E'F'}{}^{AB}(\partial a)^{E'F'}{}_{CD}. \end{aligned} \quad (133)$$

We note that this is an on-shell vertex, with all legs satisfying $\partial^\mu a_\mu^i = 0$ condition. Thus, some terms potentially relevant for loop computations have not been written down. Only two of these terms survive in the GR case, and we get

$$\begin{aligned} \mathcal{L}_{\text{GR}}^{(4)} = & \frac{\kappa^2}{M^2} (\partial a)^{ABEF} (\partial a)_{EF}{}^{CD} (\partial a)_{M'N'AB} (\partial a)^{M'N'}{}_{CD} \\ & + \frac{\kappa^2}{4M^2} (\partial a)_{M'N'}{}^{(AB} (\partial a)^{CD)M'N'} (\partial a)_{E'F'AB} (\partial a)^{E'F'}{}_{CD}. \end{aligned} \quad (134)$$

We will only write down the vertex factors corresponding to the first two terms in (133), the reason being that the other terms cannot contribute to the graviton-graviton scattering. This will become clear in the next section. Thus, in particular the GR 4-vertex present in (134) is not at relevant as far as the graviton-graviton scattering is concerned, and we have written it only for reference. It also demonstrates once again how compact the formulas become in the gauge-theoretic formalism, to be contrasted with the usual metric-based story.

For the vertex factors we will use the rule that all 4 momenta are incoming. Taking into account the symmetry factors, the associated vertices are

$$\frac{i(8g^{(4)} - 24g^{(3)} + 3g^{(2)})}{4!M^4(g^{(2)})^2} \quad \text{[Diagram: Four incoming lines labeled } k_1, k_2, k_3, k_4 \text{ meeting at a central point with four 'Sym' labels in boxes, representing a symmetric vertex factor]} \quad (135)$$

where one needs to take a sum over all possible ways to insert the 4 incoming momenta into this vertex (i.e. the full vertex involves $4!$ terms of the type shown above, some of which are of course identical). The other vertex of interest is

$$- \frac{i(8\tilde{g}^{(4)} - 8g^{(3)} + 5g^{(2)})}{3M^4(g^{(2)})^2} \quad \text{[Diagram: Four incoming lines labeled } k_1, k_2, k_3, k_4 \text{ meeting at a central point with four 'Sym' labels in boxes, representing a symmetric vertex factor]} \quad (136)$$

where now the sum over possible permutations includes only 3 terms (numbered e.g. by what k_1 gets connected to).

Just as a reference, we will also give pictures of the index contractions present in the other 3 terms appearing in (133). As we have said, these other terms will not contribute to any computations in this paper, but the pictures below will be instrumental in seeing this. We give them in the order that they appear in (133), without any associated prefactors.

$$\text{[Three diagrams showing different index contractions for the 4-point vertex, each with four incoming lines labeled } k_1, k_2, k_3, k_4 \text{ and 'Sym' labels in boxes]} \quad (137)$$

The first and the last of these are the ones that appear in (62).

8 Graviton-graviton scattering

This is the last section of the present paper, where we put to use all the technology that we developed. As is the case with the metric-based GR, even the simplest $--++$ amplitude is somewhat easier to compute as just the first non-trivial case of a more general MHV amplitude with all but two positive helicity gravitons. For such amplitudes one can write the so-called BCFW recursion relation [19], and then the $--++$ amplitude is given by a sum of two terms, both involving just the known $--+$ amplitudes (124). However, here we avoid developing the technology of recursion relations, postponing it to the next paper. Instead, we compute all amplitudes of interest by directly evaluating the relevant Feynman diagrams, of course using the spinor helicity technology that we already have developed. We will see that the $--++$ amplitude receives contributions from only tri-valent graphs with vertices (127), and there are two diagrams to compute. The degree of complexity of this computation is very similar to the analogous textbook computation in Yang-Mills theory, see e.g. [17]. We first do the computation of the $--++$ amplitude, and then consider somewhat simpler cases of the $-+++$ and $++++$ amplitudes (which are only non-zero in our more general parity-violating theories).

8.1 The all negative and all negative one positive helicity amplitudes

The amplitudes with at most one positive helicity graviton can be shown to vanish in full generality, just by a simple count of the number of derivatives present in any given diagram. This is a standard type argument, so we will be brief.

Let us first consider the all minus case. In this case the helicity states (104) each carry 3 copies of the negative helicity reference spinor q^A . This spinor can then be chosen to be the same for all the gravitons. Thus, we have $3n$ copies of q^A in the diagram, and we have to contract them all with some momenta that appear as a result of evaluating in the derivatives present in the vertices. It is then easy to see that there are not enough derivatives to avoid contracting the q^A 's between themselves. Indeed, the largest possible number of derivatives is in a diagram with 3-valent vertices only. There are $n - 2$ such vertices, and thus at most $3(n - 2)$ derivatives present (as the largest power of the derivative in each vertex is 3). Since each derivative can only eat one copy of q^A , there is not enough derivatives for these amplitudes to be non-zero.

The argument with all minus one plus proceeds similarly, but in this case one chooses all the reference spinors of the negative helicity gravitons to be the momentum spinor k^A of the positive helicity graviton. Then there are again $3n$ instances of k^A and only at most $3(n - 2)$ derivatives, which is not enough to avoid the k^A 's contracting.

8.2 The $--++$ amplitude

This is the only non-vanishing amplitude in the case of GR, as we shall also explicitly see using our methods.

Let us first verify that the 4-vertex containing diagrams cannot contribute to this amplitude. This is an argument of the same type that we already gave to show that the all minus and all minus one plus amplitudes are zero. Indeed, we now have 6 copies of q^A coming from the two negative helicity states, and we can choose them to be equal to the momentum spinor k_3^A of one of the positive helicity graviton. We thus get 9 copies of k_3^A that need to contract with something else than themselves. Let us count other object available. We have 3 copies of the other positive helicity momentum spinor k_4^A coming from its helicity spinor (104). We thus need at least 6 derivatives to absorb the remaining 6 copies of k_3^A . These can only come from tri-valent diagrams, as the other diagrams contain less derivatives. Incidentally, the same argument shows that only the 3-valent graphs contribute to any MHV amplitude (which in our convention is the amplitude with at most two plus helicity gravitons).

It remains to see that only the vertices (128) can contribute to this amplitude. Indeed, let us assume that one of the vertices used is (130). Since this vertex can only take the positive helicity gravitons,

both available positive helicity states must go in it. The other vertex thus must be necessarily (128), with two negative helicity states inserted in it. It is then a simple verification to see that in the internal edge of the diagram we will have $q^A q^B q^C$ coming from the negative helicities contracting with $k_3^{(A} k_3^B k_4^{C)}$ or with $k_3^{(A} k_4^B k_4^{C)}$ coming from the positive helicities. Since we have chosen q^A to be the momentum spinor of one of the positive helicity gravitons, this contraction is zero. As we shall see below, precisely the same argument works for a general MHV amplitude, and establishes that only the vertex (128) is relevant. This means that after the identification (129) is made, all MHV amplitudes for a general member of our family of theories are the same as in GR. In particular, the $--++$ amplitude is the same, and so in the following considerations we can assume the form (127) of the relevant vertex.

We therefore only need to consider the 3-valent graphs with vertices (128). There are 3 such graphs (s, t, u channels), and each vertex factor has 3 terms. Thus, there are in principle 27 terms to consider. However, most of them are zero.

A very convenient way to organize the computation is to consider the possible ways of putting two negative helicities into the vertex (127). It is clear that they must go into the two legs in which the vertex is symmetric. Recall that these legs came from the anti-self-dual part of the two-form da^i , so they can be referred to as the ASD legs. We can now recall our choice $q^A = k_3^A$. It means that for many purposes the positive helicity graviton of momentum k_3 behaves like a negative helicity one. In particular, if we are to put this positive helicity graviton into the vertex (127) together with some negative helicity graviton, it is easy to see that both necessarily must go into the ASD legs. Indeed, if the positive helicity goes into the bottom leg in the picture (127) it is easy to see that at least one of the q^A 's from the negative helicity graviton will contract with k_3^A along the black lines. Thus, in case of the graviton of momentum k_3 and some negative helicity graviton, both must go into the upper, ASD legs in the figure (127).

Let us now compute the result of such insertion of two negative or one negative one positive of momentum k_3 helicity states into the vertex (127). For two negative states that we label by 1 and 2, we get the following quantity

$$iM\kappa q^A q^B q^C (k_1 + k_2)^{DD'} q_D \frac{[1\,2]^2}{\langle 1\,q \rangle^2 \langle 2\,q \rangle^2}. \quad (138)$$

Computing the result of the states 1 and 3 put together we get

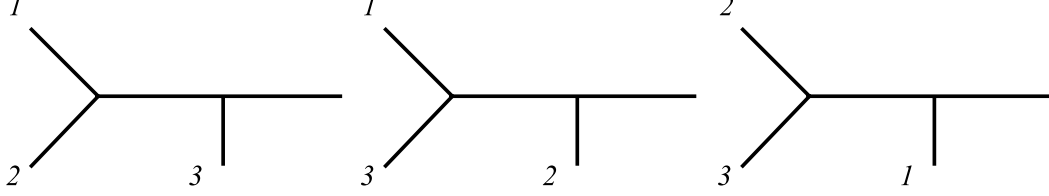
$$-iM\kappa q^A q^B q^C (k_1 + k_3)^{DD'} q_D \frac{[1\,p]^2}{\langle 1\,q \rangle^2 [3\,p]^2}, \quad (139)$$

where an extra minus is from $\partial_M^E a_{EABN'}$ where we get $k_1^E q_E = \langle 1\,q \rangle$, but in the denominator of these helicity states we have $\langle q\,1 \rangle^3$, and this different order of the contraction produces an extra minus sign. Two of such minus signs have cancelled each other in (138). We remind the reader that in (139) we have used $k_3^A = q^A$.

The important point about the results (138) and (139) is that they can now be put together in a 3-vertex (127) only in a single way, again from the upper two legs in the picture. The reason for this is that both of them, as the original helicity states, contain 3 copies of q^A , thus, the argument we gave above about the only possible way to couple such states applies. This means that the vertex (127) works only "in one direction" coupling the negative helicities or the positive helicity graviton of momentum k_1 by taking them in its ASD legs. As an aside remark, we note that this immediately implies that even in a general MHV amplitude with two plus helicity gravitons, the vertex (130) cannot be used. Indeed, it could only be used to couple the two positive helicity gravitons to all negative gravitons already coupled in some way with (128). However, the one leg off-shell current with any number of negative helicity gravitons is necessarily proportional to $q^A q^B q^C$, and this will vanish when contracted with what comes from the vertex (130), as we have already discussed above. In the next

paper we will put these facts to work and compute the one leg off-shell all negative helicity current. For now we continue our consideration of the $--++$ amplitude.

The above picture of the vertex (127) working as a coupler of states just in one direction gives just 3 possibilities to consider. One can either couple first the two negative helicity gravitons 1 and 2, and then couple the result to 3, or first couple 1 and 3, and then couple to 2, or first couple 2 to 3 and then to 1.



In all these cases one couples to the positive helicity graviton 4 at the very end. These are of course just the 3 different s, t, u channels, but we now have just 3 terms to consider instead of 27.

The final simplification comes from the availability to choose the positive helicity reference spinor $p^{A'}$ conveniently. We recall that we have not yet made any choice of this in the analysis so far, so we are now free to make the most convenient choice. We choose $p^{A'} = 2^{A'}$, which eliminates the possibility (139) to couple 3 directly to 2. This leaves just the first two terms in the above picture to consider.

Let us compute the diagram when 1 and 2 get coupled first, and then couple to 3. We have computed the result of coupling 1 and 2 in (138). We then multiply this result by the propagator $1/ik^2$, where $k^2 = k_{A'}^A k_A^{A'}$ computes, using the fact that both k_1 and k_2 are null, to just $k_{12}^2 = 2\langle 12\rangle[12]$. We now couple together (138) with the propagator added at the end with the positive helicity state 3. After applying the derivatives present in the vertex we get the following contraction

$$\begin{aligned} (i)^2 \kappa^2 q_A q_B (k_1 + k_2)^E {}_{M'} q_E (k_1 + k_2)^F {}_{N'} q_F (k_1 + k_2 + k_3)^D {}_{D'} \frac{[12]^2}{\langle 1q \rangle^2 \langle 2q \rangle^2} \frac{1}{2i\langle 12\rangle[12]} M \frac{q_C q_D p^{M'} p^{N'}}{[3p]^2} \\ = 2iM \left(\frac{\kappa}{2}\right)^2 q_A q_B q_C (k_1 + k_2 + k_3)^D {}_{D'} q_D \frac{[12]}{\langle 12\rangle} \frac{\langle q|k_1 + k_2|p\rangle^2}{\langle 1q \rangle^2 \langle 2q \rangle^2 [3p]^2}. \end{aligned}$$

We finally insert into this result the last remaining positive helicity $(1/M)4^A 4^B 4^C p^{D'}/[p4]$. We can also use the momentum conservation to replace $k_1 + k_2 + k_3$ by $-k_4$ (all momenta are incoming). Substituting also $q^A = 3^A$ and $p^{A'} = 2^{A'}$ we get, overall

$$2i \left(\frac{\kappa}{2}\right)^2 \frac{[12]^3}{\langle 12\rangle} \frac{\langle 34 \rangle^4}{\langle 23 \rangle^2 [23]^2}. \quad (140)$$

We now compute the other non-vanishing diagram, where 1 gets first connected to 3, and then the result connects to 2. Adding to (139) the propagator, taking the helicity state for 2 and applying all the derivatives gives the following contraction

$$\begin{aligned} (i)^2 \kappa^2 q_A q_B (k_1 + k_3)^E {}_{M'} q_E (k_1 + k_3)^F {}_{N'} q_F (k_1 + k_2 + k_3)^D {}_{D'} \frac{[1p]^2}{\langle 1q \rangle^2 [3p]^2} \frac{1}{2i\langle 13\rangle[13]} M \frac{q_C q_D p^{M'} p^{N'}}{\langle 2q \rangle^2} \\ = 2iM \left(\frac{\kappa}{2}\right)^2 q_A q_B q_C (k_1 + k_2 + k_3)^D {}_{D'} q_D \frac{[1p]^2}{\langle 13 \rangle [13]} \frac{\langle q|k_1 + k_3|p\rangle^2}{\langle 1q \rangle^2 \langle 2q \rangle^2 [3p]^2}. \end{aligned}$$

We now put in this the last remaining state 4, and use the values of $q^A, p^{A'}$ to get

$$2i \left(\frac{\kappa}{2}\right)^2 \frac{[12]^4}{\langle 13 \rangle [13]} \frac{\langle 34 \rangle^4}{\langle 23 \rangle^2 [23]^2}. \quad (141)$$

Adding (140) and (141) and using the momentum conservation we get

$$-2i\left(\frac{\kappa}{2}\right)^2 \frac{\langle 34 \rangle^4}{\langle 23 \rangle^2 [23]^2} \frac{[12]^3 \langle 14 \rangle [14]}{\langle 12 \rangle \langle 13 \rangle [13]}. \quad (142)$$

We now convert as many square bracket contractions into the round ones using the momentum conservation identities, e.g. $[14]/[13] = -\langle 23 \rangle / \langle 24 \rangle$. We finally get

$$\mathcal{M}^{--++} = 2i\left(\frac{\kappa}{2}\right)^2 \langle 34 \rangle^6 \frac{1}{\langle 13 \rangle \langle 14 \rangle \langle 23 \rangle \langle 24 \rangle} \frac{[12]}{\langle 12 \rangle}, \quad (143)$$

which is the usual GR result, see below.

To rewrite (143) in a more recognizable form, we evaluate the spinor contractions present in the center of mass frame, and rewrite everything in terms of the Mandelstam variables. The relevant contractions are given in (158). We get

$$\mathcal{M}^{--++} = i\left(\frac{\kappa}{2}\right)^2 \frac{s^3}{tu}, \quad (144)$$

which is the form one finds this result in e.g. [20], formula (17), modulo the factor of i that most likely has to do with different conventions, or in [21], formula (40) with the coefficient c given after formula (42). Our factors in (144) precisely match those in [21].

We emphasize that the result (144) holds for a general member of our family of theories. It is a particular case of a more general result that all MHV amplitudes (i.e. amplitudes with just two positive helicity gravitons) are the same for all members of our family, provided one identifies the Newton's constant as in (129). The generality of the result (144) is important, because it immediately tells us important information about the types of higher derivative terms that are present in the Lagrangian of our theories if one interprets them as metric theories. Indeed, for a general counterterm corrected Einstein-Hilbert Lagrangian one expects (144) to get modified, see e.g. formula (61) of [18]. The fact that this does not happen for our class of theories tells us that no such R^4 type modifications are present. Once again it illustrates a very tightly constrained character of the modifications present in our different from GR gravitational theories.

8.3 The $-+++$ amplitude

We now compute the first example of an amplitude that is zero in GR, but non-zero in a general parity-violating theory (the $+++$ amplitude encountered in the previous section is zero once the momentum is taken to be real and the momentum conservation is imposed).

Let us first check that there cannot be any contribution from the 4-valent vertices. Since we have one negative helicity, only the vertices in (137) could contribute, with the negative helicity inserted into one of the ASD legs, i.e. where the contraction of the derivative with the connection is of the form $\partial_{M'}^E a_{EABN'}$. However, such a leg gets necessarily contracted with another ASD leg, where we will have a positive helicity graviton. As usual, we can choose the reference momenta of all positive helicity gravitons to be the same and equal to the momenta of the negative one. Let the negative helicity graviton be of momentum k_1 , then we choose $p^{A'} = 1^{A'}$. It is now easy to see that the primed index contraction will give that of $p^{A'}$ with itself, and so these 4-valent graph diagrams cannot contribute.

It remains to compute contributions from the 3-valent graphs. There are two types of vertices that we have, namely (128) and (130). Since there is one negative helicity we cannot have just twice the (130) vertex. Thus, it is either twice the usual GR vertex (128), or a combination of (128) and (130).

It is easy to check that the diagram with twice the GR vertex gives a zero result (it should, since in GR this amplitude is zero). But let us verify this in our formalism. The argument starts in the same way as when we argued that there are no contributions from the 4-valent graphs. Indeed, the

negative helicity graviton will have to be inserted into one of the ASD legs of the vertex (128), i.e. from the top in that picture. If the positive helicity graviton is also inserted into the remaining ASD leg, this will give (twice) a contraction of $1^{A'}$ with $p^{A'}$ along the red lines, which is zero because of our choice $p^{A'} = 1^{A'}$. Thus, the only option is that the negative helicity graviton gets inserted into one of the top legs, and the positive helicity is inserted from the bottom. Then it is easy to see that on the red line of the free ASD leg one will have the spinor $1^{A'}$. This should be contracted with what appears from the other vertex, where two positive helicities are combined. It is easy to see, by the same argument as we used above, that the two positive helicities cannot be inserted both into the ASD legs, and one of them must go into the bottom leg in the figure (128). This means that on the free leg of the vertex combining the two positive helicities one will again obtain $p^{A'}$ on the red line. This will contract with $1^{A'}$ from the other vertex and give zero.

The only possibly non-vanishing diagram is therefore that involving one GR (128) and one new (130) vertex. We must necessarily insert the negative graviton into the vertex (128), in one of the upper legs. We already know that some positive helicity graviton should go into the bottom leg. Let us compute the corresponding contraction, including the propagator at the end. We take the positive graviton to be the one of momentum k_2 , and get

$$\begin{aligned} & -\frac{i\kappa}{M} M \frac{q_E q_F 1^{M'} 1^{N'}}{\langle 1 q \rangle^2} \frac{1}{M} 2^E 2^F 2^{(A_2 B)} (k_1 + k_2)^C{}_{N'} \frac{1}{2i \langle 1 2 \rangle [1 2]} \\ & = \frac{\kappa}{2M^2} \frac{\langle 2 q \rangle^2}{\langle 1 2 \rangle \langle 1 q \rangle^2} 2^{A_2 B C} 1^{M'}. \end{aligned} \quad (145)$$

The free indices here are $ABCM'$. We have to contract this object with one obtained by combining the gravitons 3 and 4 via the vertex (130). This gives

$$\frac{i\sqrt{2}(4g^{(3)} - 3g^{(2)})}{M^2(g^{(2)})^{3/2}} \frac{1}{M^2} 3^E 3^F 4_E 4_F 3_{(A} 3_B 4_C 4_{D)} (k_3 + k_4)^D{}_{M'}. \quad (146)$$

The symmetrization here, together with its contraction with $k_3 + k_4$ can be written as

$$\begin{aligned} 3_{(A} 3_B 4_C 4_{D)} (k_3 + k_4)^D{}_{M'} &= \frac{1}{2} (3_D 3_{(A} 4_B 4_{C)} + 4_D 4_{(A} 3_B 3_{C)}) (k_3 + k_4)^D{}_{M'} \\ &= \frac{1}{2} (\langle 4 3 \rangle 3_{(A} 4_B 4_{C)} 4_{M'} + \langle 3 4 \rangle 4_{(A} 3_B 3_{C)} 3_{M'}). \end{aligned} \quad (147)$$

Overall, after contracting the above two quantities, we get for this contribution to the amplitude

$$\frac{i(4g^{(3)} - 3g^{(2)})}{2M^3 M_p^3} \frac{\langle 2 q \rangle^2 \langle 3 4 \rangle^3 \langle 2 3 \rangle \langle 2 4 \rangle}{\langle 1 2 \rangle \langle 1 q \rangle^2} (\langle 2 4 \rangle [1 4] - \langle 2 3 \rangle [1 3]). \quad (148)$$

This is 34 symmetric, as it should be. Note that we have replaced κ with $\sqrt{2}/M_p$ and also used the fact that $M^2 g^{(2)} = M_p^2$. We should now add contributions from the 2 more diagrams in which 1 first gets connected to 3 and then to 1 and 4, and another one where 1 first gets connected to 4 and then to the rest. What one gets can be checked to be q -independent, as it should be of course. So, we shall make a choice and set $q^A = 4^A$, so that there is just one more contribution to consider. It reads

$$\frac{i(4g^{(3)} - 3g^{(2)})}{2M^3 M_p^3} \frac{\langle 3 q \rangle^2 \langle 2 4 \rangle^3 \langle 3 2 \rangle \langle 3 4 \rangle}{\langle 1 3 \rangle \langle 1 q \rangle^2} (\langle 3 4 \rangle [1 4] - \langle 3 2 \rangle [1 2]). \quad (149)$$

We now set $q^A = 4^A$ and add the above two quantities. We can also use the momentum conservation to note that the quantities in brackets in both (148) and (149) are equal, so that we can keep only the first one of them in each case, and double the result. We thus get

$$\frac{i(4g^{(3)} - 3g^{(2)})}{M^3 M_p^3} [1 4] \frac{\langle 2 4 \rangle^3 \langle 3 4 \rangle^3 \langle 2 3 \rangle}{\langle 1 4 \rangle^2} \left(\frac{\langle 2 4 \rangle}{\langle 1 2 \rangle} - \frac{\langle 3 4 \rangle}{\langle 1 3 \rangle} \right). \quad (150)$$

Using the Schouten identity this finally gives

$$\mathcal{M}^{-+++} = i \frac{(4g^{(3)} - 3g^{(2)})}{M^3 M_p^3} \frac{[14]}{\langle 12 \rangle \langle 13 \rangle \langle 14 \rangle} \langle 24 \rangle^3 \langle 34 \rangle^3 \langle 23 \rangle^2. \quad (151)$$

Using the momentum conservation this can be seen to be 234 symmetric.

It is instructive to rewrite the result (151) using the Mandelstam variables. The relevant spinor contractions in the center of mass frame are given in the Appendix, see (158). One gets

$$\mathcal{M}^{-+++} = \frac{(4g^{(3)} - 3g^{(2)})}{8iM^3 M_p^3} stu. \quad (152)$$

Let us now discuss the important question of when the amplitude (152) becomes important. We see that it goes as $(4g^{(3)} - 3g^{(2)})E^6/M^3 M_p^3$. One would naively say that it becomes important at very low energies $E \sim \sqrt{MM_p}$, which, given that M is the extremely low energy scale associated with the cosmological constant, would contradict observations. However, the naive estimate ignores that fact that at low energies gravity should be described by GR, and so the difference $(4g^{(3)} - 3g^{(2)})$ should go to zero at low energies. The way that this happens is presumably controlled by some renormalization group flow, which we have nothing to say about at the moment. In the absence of any information about the behavior of $(4g^{(3)} - 3g^{(2)})$ we just design the defining function of our theory so that this difference goes to zero as M^3/M_p^3 as $M \rightarrow 0$. We are free to do this because the couplings $g^{(3)}$ and higher are in our disposal, and can be chosen as we wish, so we simply make the choice convenient for us. With this choice the amplitude (152) will go as E^6/M_p^6 , and thus be very small at ordinary sub-Planckian energies. The question of whether this choice of the couplings is physically realistic can only be answered if the renormalization group flow controlling the behavior of the defining function at low energies is understood, and this is beyond the scope of the present paper.

8.4 The + + + + amplitude

We now study the final graviton-graviton amplitude, involving 4 incoming gravitons of positive helicity. This amplitude vanishes if there are just the GR vertices, but is non-vanishing in general, as we shall now compute.

Let us first check that the 4-vertices in (137) do not contribute. This is a version of the argument we have already seen several times in the previous subsection. Whenever the positive helicity hits an ASD leg of a vertex, one obtains two factors of $p^{A'}$ reference spinor. These will then necessarily get contracted (along the red lines in our pictures) to similar factors from the other leg of the vertex. The only way to avoid this happening is to use the vertices that have no ASD legs at all, which are (135) and (136).

Let us discuss contributions from the 3-valent diagrams. One cannot get any non-zero result when both vertices are those of GR. Indeed, we have verified in the previous subsection that in this case one would get two copies of $p^{A'}$ contracting via the propagator. There is a non-zero contribution from the diagram with one GR vertex and one of the type (130). However, one can do a quick calculation of the dependence of this contribution to the amplitude on the energy. One easily sees that it goes as $(4g^{(3)} - 3g^{(2)})E^6/M^2 M_p^4$. In our discussion of the behavior of the difference $(4g^{(3)} - 3g^{(2)})$ in the previous subsection we have assumed that it goes as M^3/M_p (in order for the amplitude $- + + +$ to have a well-defined Minkowski limit). With this assumption the contribution to the + + + + amplitude that we are now discussing goes to zero as $M \rightarrow 0$, and so we shall not compute it in details. However, this is a straightforward calculation of the type given in the previous subsection.

Let us now discuss the contribution from the remaining 3-valent diagram with both vertices of the type (130). This is again non-zero, and a quick count of the parameters present shows that it goes as $(4g^{(3)} - 3g^{(2)})^2 E^8/M^2 M_p^6$. With our assumption that $(4g^{(3)} - 3g^{(2)}) \sim M^3/M_p$ this goes to zero as

$M \rightarrow 0$, and we shall not consider this contribution anymore. Again, if necessary, it is very easy to compute.

We thus have to consider only the contributions from the 4-valent graphs with vertices (135) and (136). The computation is easy, because after the derivatives get applied to the external states, one is just left with $(1/M)k^A k^B k^C k^D$ quantities, where k^A is the corresponding momentum spinor, to contract as dictated by the black lines in the figures (135) and (136). For the vertex (135) we get

$$\frac{i(8g^{(4)} - 24g^{(3)} + 3g^{(2)})}{3M^8(g^{(2)})^2} (\langle 12 \rangle^2 \langle 23 \rangle^2 \langle 34 \rangle^2 \langle 41 \rangle^2 + \langle 12 \rangle^2 \langle 24 \rangle^2 \langle 43 \rangle^2 \langle 31 \rangle^2 + \langle 13 \rangle^2 \langle 32 \rangle^2 \langle 24 \rangle^2 \langle 41 \rangle^2)$$

and for the vertex (136) the result is

$$- \frac{i(8\tilde{g}^{(4)} - 8g^{(3)} + 5g^{(2)})}{3M^8(g^{(2)})^2} (\langle 13 \rangle^4 \langle 24 \rangle^4 + \langle 12 \rangle^4 \langle 34 \rangle^4 + \langle 14 \rangle^4 \langle 23 \rangle^4).$$

The full scattering amplitude (with the assumptions we made about the other contributions) is the sum of the above two quantities.

We can rewrite the above results more compactly by going into the center of mass frame and introducing the Mandelstam variables (159). Using the results of spinor contractions given in (158) we get for the first contribution

$$\frac{i(8g^{(4)} - 24g^{(3)} + 3g^{(2)})}{2^4 3 M^8 (g^{(2)})^2} (s^2 u^2 + s^2 t^2 + t^2 u^2), \quad (153)$$

and for the second

$$- \frac{i(8\tilde{g}^{(4)} - 8g^{(3)} + 5g^{(2)})}{2^4 3 M^8 (g^{(2)})^2} (t^4 + s^4 + u^4). \quad (154)$$

We can now use the fact that $s^2 t^2 + s^2 u^2 + t^2 u^2 = (s^4 + t^4 + u^4)/2$ to add the above contributions to get

$$\mathcal{M}^{++++} = \frac{i(g^{(4)} - 2\tilde{g}^{(4)} - g^{(3)} - (7/8)g^{(2)})}{12M^6 M_p^2} (s^4 + t^4 + u^4). \quad (155)$$

We see that in order for this to have a well-defined Minkowski limit we need to assume that $(g^{(4)} - 2\tilde{g}^{(4)} - g^{(3)} - (7/8)g^{(2)})$ goes to zero as M^6/M_p^6 . This is of course possible by choosing $g^{(4)}$ and $\tilde{g}^{(4)}$ appropriately, and we make this choice. The above amplitude then goes as E^8/M_p^8 at high energies.

9 Discussion

The main outcome of this work is a derivation of the Feynman rules, as well as of the prescription for computing the Minkowski space graviton scattering amplitudes. For the convenience of the reader these are listed in the Appendix.

Our main physical result is the computation of the graviton-graviton scattering amplitudes. We recall that in GR there is just one non-vanishing such amplitude (in the convention that all particles are e.g. incoming), and this is the amplitude with two gravitons of one and two of the opposite helicity. Using the terminology of maximally helicity violating (MHV) amplitudes (in our conventions these are amplitudes with just two positive helicities), the GR graviton-graviton amplitudes is the simplest MHV amplitude. Our computation confirms the GR result, which is completely expected since we work with just a different, but equivalent at the tree level formulation of general relativity.

However, some aspects of this computation deserve to be emphasized. As we already stressed in the Introduction, the gauge-theoretic formulation of GR is quite different from the metric-based one,

with one of the most significant distinctive features being that the graviton conformal mode does not propagate even off-shell. The fact that such a reformulation is possible does not come as a surprise. Indeed, in the formulation due to Bern, see e.g. [20], one introduces an additional scalar field, and then does a field redefinition of the metric variable in such a way that the conformal mode of the graviton completely disappears (in favor of the new scalar field). Then, for the purposes of computing the tree-level scattering amplitudes the scalar field can be forgotten about completely, because it is not sourced on the external lines, and thus cannot propagate on the internal lines either. What happens in our formulation is similar, except that the statement that the conformal mode does not propagate is true in even greater generality, as it extends to loops. Thus, the gauge-theoretic formulation can be expected to start to differ from the metric-based GR when one computes quantum effects. And at the tree level, our formulation automatically produces the simple 3-vertex (119) which is at the heart of the simplicity of the present formalism. While many simplifications of the metric-based GR cubic vertex are possible by a clever choice of gauge-fixing as in e.g. [22], or by the trick of introducing an additional scalar field as in [20], our formulation produces the simplest imaginable, as well as most practical for calculations expression for this vertex automatically, without too much thought.

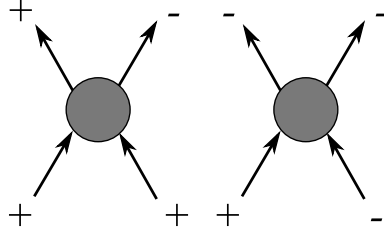
We have also stressed a very strong analogy between the linearized Lagrangian, as well as the cubic interaction in our gauge-theoretic formulation of GR and Yang-Mills. As we already discussed after introducing the cubic interaction in (119), both the linearized Lagrangian and the cubic interaction in gauge-theoretic gravity are more or less direct generalizations of the corresponding objects in Yang-Mills theory. The generalization in question is the passage from an object with one unprimed one primed spinor index $A_{MM'}$, which is the Yang-Mills spin 1 field with its color index suppressed, to an object $A_{ABMM'}$ with 3 unprimed indices and one primed, which is the $SU(2)$ gauge field that we used for our description of gravitons.

We now come to the discussion of even more intriguing aspects of the present formalism. One of the most dramatic outcomes of the studied here (and related) reformulations of GR is that general relativity is not the only theory of interacting massless spin 2 particles. Indeed, the uniqueness of GR as the only possible theory of interacting gravitons is taken for granted by many, and became a part of our intuition about gravity. And still what we have been studying here is a very concrete set of counterexamples of the GR uniqueness statement. Indeed, we have studied a large (parameterized by an infinite number of coupling constants) class of gravitational theories, all describing just two propagating degrees of freedom. They were treated perturbatively, thus as interacting theories of gravitons. All these theories were seen to share the same linearized dynamics, which is also coincides with the linearization of GR reformulated in this language. Thus, very concretely, all theories considered here are interacting theories of gravitons, with interactions of other members of our family being distinct from those in GR.

In this paper we have been able to study these different graviton interactions quantitatively. We have seen that, most importantly, the new interactions are parity-violating, in that they lead to parity violating scattering amplitudes. Thus, it appears likely that *if* the condition of parity-invariance is added, then GR may indeed be the unique theory of interacting gravitons. We have not attempted to prove any such statement here, but since all the deviations from GR that we have found are in the direction of parity violation, it is possible that such GR uniqueness statement can be shown to hold. However, there is no physical reason to restrict one's attention to only parity-invariant theories. Indeed, we know that Nature does violate parity invariance (in the Standard Model) in the strongest possible way. It is gratifying to see that gravitational theories where parity invariance is not built in from the start are also possible.

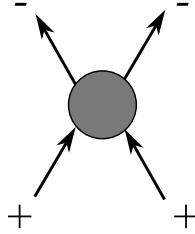
Let us now discuss the obtained parity-violating amplitudes in more details. First, there is a set of amplitudes where parity is flipped just by one unit. Applying the crossing symmetry and converting

two gravitons from incoming to outgoing states, we can draw these parity-violating processes as



In both cases one of the positive helicity gravitons is converted into a negative helicity one. We have seen (152) that this amplitude goes as stu/M_p^6 , where s, t and u are the usual Mandelstam variables. It is thus of equal importance as the usual parity-preserving amplitude $(1/M_p^2)s^3/tu$ at Planck energies, which is just an illustration of the fact that a generic member of our family of gravity theories is very different from GR at high energies. If one could extrapolate beyond the Planck barrier (which in reality one cannot because the perturbation theory breaks down), one could say that at higher energies the parity-violating processes scaling as E^6 become even more important than the parity-preserving ones scaling as E^2 .

Another set of the processes is when the helicity is flipped by two units. This can be drawn as



Two positive helicity gravitons are converted by this process into two negative helicity ones, with the amplitude computed in (155) going as $(s^4 + t^4 + u^4)/M_p^8$. As the amplitude now scales as E^8 , it clearly costs more energy to perform a double helicity flip as compared to single one, but at Planck energies all these processes become equally important.

One is then led to admittedly speculative, but thought-provoking picture of the dynamics of gravity at high energies, as predicted by our theories. Indeed, the parity-violating processes only go in one direction (the theories are not T-invariant). It is then clear that eventually all gravitons will get converted into gravitons of a single helicity (negative in our conventions). Whether this indeed has anything to do with what happens at the Planck scale remains to be seen, but it is clear that this picture is a valid, and to some extent unexpected outcome of our gauge-theoretic approach.

To conclude the discussion of the parity-violating amplitudes let us comment on how our theories manage to avoid the proof [21] of GR uniqueness that works directly with the scattering amplitudes. One point about the work [21] that makes it not general enough is the assumption of the parity symmetry. This is clearly violated by our theories. The absence of parity is also the reason why in our case it is possible to get non-zero answers for parity-violating processes, while in [21] parity can be easily shown to lead to vanishing of the $2, 2; -2, -2$ amplitude, in the conventions of these authors. The vanishing of another amplitude $2, 2; 2, -2$ is shown in [21] by a more subtle argument, which does not relate to parity in a direct way. The argument is that the kinematical singularities of this amplitude are known, and then the non-singular piece must behave (for dimensional reasons) as such a high power of momentum that this is impossible in a theory with at most two derivatives in a vertex. The way this argument is circumvented is instructive. While our theories lead to second-order in derivatives field equations, they have arbitrary number of derivatives at vertices when expanded, in the sense that the n -th order vertex can contain as many as n derivative, but always with just a single derivative acting on each leg of the vertex. This is the reason why the argument in [21] for the

case of the $2, 2; 2, -2$ amplitude is not applicable to our theories, and indeed, we see a non-vanishing single helicity flip amplitude (152).

One unsatisfactory aspect of our analysis is our inability to control the form of the defining function of the theory, e.g. as a function of the energy scale. Thus, in contrast to the metric approach where the Einstein-Hilbert action is clearly the most dominant term at low energies, we do not at present understand why at low energies the defining function of our theory should be chosen the way it is for GR. From our results we can only see that the scattering amplitudes for parity violating processes go as much higher power of energy than is the case for GR amplitudes. This does indicate that at low energies these processes can be neglected, which to some extent explains why the parity-preserving GR is the low energy theory. But it would be nice to convert this qualitative argument into a quantitative, and this is missing at the moment.

Another, related aspect is our lack of understanding of the renormalizability properties of our class of theories. The question is whether any counterterms that are not already present in the Lagrangian need to be added in the process of renormalization. If, as originally conjectured in [23], it happens that this class of theories is closed under the renormalization, one could compute the associated RG flow, and then answer the question of how the defining function flows with energy in a quantitative way. In the absence of any such result, we could only choose the defining function by hand, which is what we did in obtaining the above stated amplitude results. In particular, our conclusion that the parity-violating processes are of importance only at Planck energies was based on such choices for the defining function. If in reality the defining function does not approach the GR one at low energies as we assumed in the main text, then the energy scales where the parity-violating processes become important may be different. In particular, because the cosmological constant $\Lambda = M^2$ gets involved, see (152), (155), the associated energy scales may be much lower than M_p , which is a potentially very interesting effect.

One other important outcome of this work is a better understanding of the modification of GR that is provided by a generic member of our family of theories. We have already discussed the parity-violating aspect of this modification. Another aspect so far less emphasized is the fact that the graviton-graviton parity-preserving amplitude is unmodified at all, and is the same for all members of our family. An even stronger statement is true: All MHV amplitudes, defined as amplitudes with just two positive helicity gravitons, are exactly the same as in GR. We have sketched an argument to this effect in the main text, and will study these amplitudes in more details in the next paper. Let us concentrate on the graviton-graviton case. A general metric theory with higher powers of curvature present in the Lagrangian will modify the graviton-graviton amplitude, basically due to the fact that it introduces new (higher derivative) vertices. The fact that there is no such modification taking place for all our theories tells us a lot about them. In the next paper we will see that this has to do with the second-derivative nature of our class. We shall see that several very general statements are possible for second-derivative gravity theories, and, in particular, we will see that the graviton-graviton parity-preserving amplitude cannot be anything else but what it is in GR. Thus, it is not too surprising that this particular amplitude does not get modified. But it is worth emphasizing that this signals once more that the theories considered are *not* the most general gravity theories, the latter tending to introduce additional propagating degrees of freedom and, in particular, modify the graviton-graviton amplitude.

Another very interesting aspect of the modification provided by our theories is that (at tree level) they continue to give zero results for the mostly negative helicity amplitudes, such as amplitudes with all gravitons of negative helicity, or just with a single positive helicity particle. This is of course the case for GR, but it continues to hold for all members of the class we have studied. This property can in turn be shown to be related to the property of MHV amplitudes being unmodified. And this property can again be used to characterize the type of corrections present in our theories as compared to GR. Indeed, a general metric Lagrangian quantum corrected by all curvature invariants will typically render both all minus and all plus amplitudes non-zero. For example, even the simplest term Riemann

curvature cubed (the Goroff-Sagnotti term) that needs to be added as a counterterm at two loops will give rise to non-zero answers for both the all negative and the all positive graviton-graviton amplitude. The fact that this does not happen in our case tells us that the type of counterterm corrections that have been included in our theories on top of what is present in GR is very special. Indeed, as we have already discussed in the text, this means that only the self-dual part of the Weyl curvature cubed is present in our Lagrangian, rather than the full Weyl cubed. It may thus seem that our class of theories has no chance of being closed under the renormalization, as containing not the full Goroff-Sagnotti term, but rather only its chiral half. While this is a legitimate worry, we also recall that our theories are expected to differ from GR as far as quantum corrections are concerned (being equivalent only on-shell). Thus, it is possible that their two-loop behavior is different, and that only the self-dual part of Weyl cubed arises as a counterterm. More work is needed to see if this is the case. However, it is interesting that results in the present paper suggest yet another clear test of how the closedness under the renormalization can be probed (by performing the two-loop computation [13] for GR but in the gauge-theoretic language).

We close by stating once more our amazement at how efficient and powerful the presented gauge-theoretic formulation of gravity is for practical computations. We did not expect this amount of simplicity when we embarked on the present study. It also comes to us as a surprise that the present family of gravity theories can be understood in rather complete generality, and many statements that are true for all members of the family are possible. In the next paper from this series we will see even more instances of this, in that it will be possible to characterize all tree level scattering amplitudes, not just MHV, to certain extent. The studied here reformulation of gravity as a diffeomorphism invariant gauge theory thus continues to be a source of fascination.

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Appendix: Center of mass frame momentum spinors and Mandelstam variables

The center of mass frame expressions for our momentum spinors can be obtained from (98). We take the particles 1, 2 to be moving in the direction of the \vec{z} axes, positive and negative respectively. This gives

$$1^A = 2^{1/4} \sqrt{\omega_k} o^A, \quad 2^A = i 2^{1/4} \sqrt{\omega_k} \iota^A. \quad (156)$$

The particles 3, 4 we take to be the scattered ones, moving at an angle θ to the \vec{z} axes. For simplicity we put $\phi = 0$. We get

$$3^A = 2^{1/4} \sqrt{\omega_k} (\sin(\theta/2) \iota^A + \cos(\theta/2) o^A), \quad 4^A = i 2^{1/4} \sqrt{\omega_k} (\sin(\theta/2) o^A - \cos(\theta/2) \iota^A). \quad (157)$$

The non-zero contractions are

$$\begin{aligned} \langle 1 2 \rangle &= -i\sqrt{2}\omega_k, & \langle 1 3 \rangle &= -\sqrt{2}\omega_k \sin(\theta/2), & \langle 1 4 \rangle &= i\sqrt{2}\omega_k \cos(\theta/2), \\ \langle 2 3 \rangle &= i\sqrt{2}\omega_k \cos(\theta/2), & \langle 2 4 \rangle &= -\sqrt{2}\omega_k \sin(\theta/2), & \langle 3 4 \rangle &= i\sqrt{2}\omega_k. \end{aligned} \quad (158)$$

It is also customary to introduce the following Mandelstam variables

$$s = -4\omega_k^2, \quad t = 4\omega_k^2 \sin^2(\theta/2), \quad u = 4\omega_k^2 \cos^2(\theta/2). \quad (159)$$

Appendix: Feynman rules

In this Appendix we collect all the Feynman rules derived in the main text, directly in spinor notations that are most convenient for practical computations.

In spinor notations the vertices involve only the factors of momentum spinors $k^{AA'}$ on the legs of the vertex, as well as factors of spinor metrics $\epsilon^{AB}, \epsilon^{A'B'}$. Writing down the corresponding expressions can quickly lead to horribly looking formulas. For this reason it is much more efficient to draw the vertices, indicating the factors of $\epsilon^{AB}, \epsilon^{A'B'}$ by lines. The only drawback of this procedure is that it is not easy to keep track of the signs (remember that raising-lowering a pair of spinor indices induces a minus sign). We have not tried to develop any convention for these signs, just going back to the corresponding term in the Lagrangian and seeing how the indices contract when there is a question. But it is possible that a more systematic sign convention can be developed. Here, in view of the fact that only simple computations are done, our rules are sufficient.

With these remarks in mind, let us state the rules of the game. First, we only state here the rules of computing the Minkowski space amplitudes. The way these are obtained as a limit of more general de Sitter graviton amplitudes is explained in the main text. The field that propagates in our theory is an $SU(2)$ connection, but after all the gauge-fixings and translation into the spinor notations, this is a field $a_{ABCC'}$ with 4 spinor indices, 3 unprimed and one primed. It is moreover symmetric in its 3 unprimed spinor indices, thus forming an object that takes values in an irreducible representation of the Lorentz group. Thus, only $4 \times 2 = 8$ components of the field propagate, as compared to 10 in the usual metric treatment. As in any textbook example, the scattering amplitudes are obtained from the field (connection) correlation functions by certain reduction formulas. These are most practical in the momentum space representation, where the correlation function is that of the Fourier coefficients of the field operator. The scattering amplitude then reads

$$\langle k_- \dots k_+ \dots | p_- \dots p_+ \dots \rangle = (-1)^{m+m'} (2\pi)^4 \delta^4 \left(\sum k - \sum p \right) \quad (160)$$

$$\langle T \varepsilon^+(k_-) \cdot a(k_-) \dots \varepsilon^-(k_+) \cdot a(k_+) \dots \varepsilon^-(p_-) \cdot a(p_-) \dots \varepsilon^+(p_+) a(p_+) \rangle_{\text{amp}},$$

where we denoted the contractions of the spinor indices involved by a dot, in other words $\varepsilon(k) \cdot a(k) \equiv \varepsilon_{ABCC'}(k) a^{ABCC'}(k)$, and k_-, k_+ are the momenta of the outgoing negative and positive helicity gravitons, p_-, p_+ are those of the incoming particles, and m, m' are the numbers of positive helicity outgoing and incoming gravitons respectively. As usual, the momentum space amplitude is amputated from its external line propagators, and there is a factor of the total momentum conservation.

The quantities ε^\pm are the graviton helicity states that read

$$\varepsilon_{ABCA'}^-(k) = M \frac{q_A q_B q_C k_{A'}}{\langle q k \rangle^3}, \quad \varepsilon_{ABCA'}^+(k) = \frac{1}{M} \frac{k_A k_B k_C p_{A'}}{[p k]}, \quad (161)$$

where M is the mass scale of the background de Sitter space, and $q^A, p^{A'}$ are the negative and positive helicity reference spinors, which we denote (for convenience) by different letters.

We have the usual statement of the crossing symmetry, which is that one can change an outgoing state into an incoming one, if one flips the direction of the arrow on the corresponding external leg, and flips the helicity. In addition, there is a factor of minus sign for any such flip, but this is a result of our convention choices. Because of the crossing symmetry, one can assume all particles to be e.g. incoming, which is what we do.

Finally, the rule is that all positive helicity (incoming) particles are taken to be slightly massive, with the mass related to the mass scale M in (161). The meaning of k_A in this formula is then explained by the following decomposition of the 4-momentum $k^{AA'}$

$$k^{AA'} = k^A k^{A'} + M^2 \frac{p^A p^{A'}}{\langle p k \rangle [p k]}. \quad (162)$$

The convention is that the reference spinors $p^A, p^{A'}$ in this formula are the same as what is used in the positive helicity spinor in (161). The negative helicity particles are all massless.

The propagator of the theory is best represented as a drawing, consisting of a set of lines contracting the indices, and a black box denoting the symmetrization of the unprimed spinor indices. Black lines represent unprimed indices, while the red line is for the single primed index. The propagator then reads

$$\frac{1}{ik^2} \quad \begin{array}{c} \longrightarrow \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array}$$

In the same graphic notation, the 3-valent vertex that is relevant for the computation of the MHV amplitudes (we referred to it as parity-preserving vertex) reads

$$\frac{i\kappa}{M} \quad \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \quad , \quad (163)$$

where $\kappa^2 = 32\pi G$. For GR this is the only relevant vertex (at tree level, for MHV amplitudes). The general member of our family of theories contains an additional 3-vertex, which reads

$$i\kappa^3 M(2g^{(3)} - (3/2)g^{(2)}) \quad \begin{array}{c} \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \\ \text{---} \text{---} \text{---} \text{---} \text{---} \text{---} \end{array} \quad (164)$$

where $(2g^{(3)} - (3/2)g^{(2)})$ is a certain combination of the coupling constants that vanishes in the case of GR. Drawings for the 4-valent vertices can be found in the main text.

References

- [1] K. Krasnov, “Pure Connection Action Principle for General Relativity,” *Phys. Rev. Lett.* **106**, 251103 (2011) [arXiv:1103.4498 [gr-qc]].
- [2] K. Krasnov, “Gravity as a diffeomorphism invariant gauge theory,” *Phys. Rev. D* **84**, 024034 (2011) [arXiv:1101.4788 [hep-th]].
- [3] R. Capovilla, T. Jacobson and J. Dell, “General Relativity Without the Metric,” *Phys. Rev. Lett.* **63**, 2325 (1989).
- [4] R. Capovilla, T. Jacobson and J. Dell, “A Pure spin connection formulation of gravity,” *Class. Quant. Grav.* **8**, 59 (1991).
- [5] R. Capovilla, “Generally covariant gauge theories,” *Nucl. Phys. B* **373**, 233 (1992).
- [6] I. Bengtsson, “The Cosmological constants,” *Phys. Lett. B* **254**, 55 (1991).
- [7] K. Krasnov, “Effective metric Lagrangians from an underlying theory with two propagating degrees of freedom,” *Phys. Rev. D* **81**, 084026 (2010) [arXiv:0911.4903 [hep-th]].

- [8] G. Delfino, K. Krasnov and C. Scarinci, “Pure Connection Formalism for Gravity: Linearized Theory,” arXiv:1205.7045 [hep-th].
- [9] E. Witten, “Perturbative gauge theory as a string theory in twistor space,” Commun. Math. Phys. **252**, 189 (2004) [arXiv:hep-th/0312171].
- [10] R. Penrose and W. Rindler, “SPINORS AND SPACE-TIME. 1. TWO SPINOR CALCULUS AND RELATIVISTIC FIELDS,” *Cambridge, Uk: Univ. Pr. (1984) 458 P. (Cambridge Monographs On Mathematical Physics)*
- [11] K. Krasnov, “A Gauge Theoretic Approach to Gravity,” Proc. Roy. Soc. Lond. A **468**, 2129 (2012) [arXiv:1202.6183 [gr-qc]].
- [12] D. Marolf, I. A. Morrison and M. Srednicki, “Perturbative S-matrix for massive scalar fields in global de Sitter space,” arXiv:1209.6039 [hep-th].
- [13] M. H. Goroff and A. Sagnotti, “The Ultraviolet Behavior of Einstein Gravity,” Nucl. Phys. B **266**, 709 (1986).
- [14] P. De Causmaecker, R. Gastmans, W. Troost and T. T. Wu, “Helicity Amplitudes for Massless QED,” Phys. Lett. B **105**, 215 (1981).
- [15] J. F. Gunion and Z. Kunszt, “Improved Analytic Techniques for Tree Graph Calculations and the $G g q$ anti- q Lepton anti-Lepton Subprocess,” Phys. Lett. B **161**, 333 (1985).
- [16] G. Chalmers and W. Siegel, “Simplifying algebra in Feynman graphs, Part I: Spinors,” Phys. Rev. D **59**, 045012 (1999) [arXiv:hep-ph/9708251].
- [17] M. Srednicki, “Quantum field theory,” Cambridge, UK: Univ. Pr. (2007) 641 p
- [18] N. Arkani-Hamed, F. Cachazo and J. Kaplan, “What is the Simplest Quantum Field Theory?,” JHEP **1009**, 016 (2010) [arXiv:0808.1446 [hep-th]].
- [19] R. Britto, F. Cachazo, B. Feng and E. Witten, “Direct proof of tree-level recursion relation in Yang-Mills theory,” Phys. Rev. Lett. **94**, 181602 (2005) [hep-th/0501052].
- [20] Z. Bern, “Perturbative quantum gravity and its relation to gauge theory,” Living Rev. Rel. **5**, 5 (2002) [gr-qc/0206071].
- [21] M. T. Grisaru, P. van Nieuwenhuizen and C. C. Wu, “Gravitational Born Amplitudes and Kinematical Constraints,” Phys. Rev. D **12**, 397 (1975).
- [22] A. E. M. van de Ven, “Two loop quantum gravity,” Nucl. Phys. B **378**, 309 (1992).
- [23] K. Krasnov, “Renormalizable Non-Metric Quantum Gravity?,” hep-th/0611182.